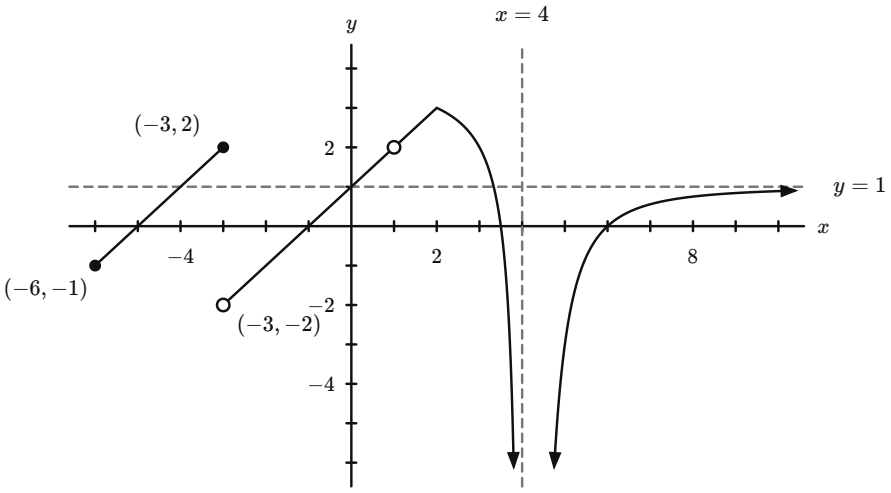
Calculus 1 Unit 1 worksheet: What a limit is: graphs and tables (answer key)

ANSWERS AT A GLANCE

1. (a) $\lim_{x\to -6^+} g(x) = -1$; g has no inputs to the left of −6, so only the right-hand limit can be asked. (b) 2, −2, does not exist, and $g(-3) = 2$. (c) $\lim_{x\to 1} g(x) = 2$, and $g(1)$ is undefined. (d) $\lim_{x\to 4^-} g(x) = -\infty$ and $\lim_{x\to 4^+} g(x) = -\infty$, so $\lim_{x\to 4} g(x) = -\infty$; there is no such number L, the limit does not exist. (e) $\lim_{x\to\infty} g(x) = 1$.
2. (a) $S_1 = \frac{5}{3}$, $S_2 = \frac{5}{2}$, $S_3 = 3$, $S_{98} = 4.9$, $S_{998} = 4.99$. (b) $\lim_{n\to\infty} S_n = 5$, since the gap $5 - S_n = \frac{10}{n+2}$ shrinks toward 0; the sum of the series is 5. (c) $a_1 = \frac{5}{3}$, $a_2 = \frac{5}{6}$, $a_3 = \frac{1}{2}$; the ratios $\frac{a_2}{a_1} = \frac{1}{2}$ and $\frac{a_3}{a_2} = \frac{3}{5}$ differ, so the series is not geometric.
3. (a) $f(-0.1) = 2.86797$, $f(-0.01) = 2.73200$, $f(-0.001) = 2.71964$, $f(0.001) = 2.71692$, $f(0.01) = 2.70481$, $f(0.1) = 2.59374$. (b) The left outputs fall and the right outputs rise toward the same number; $\lim_{x\to 0} f(x) \approx 2.718$. (c) The number $e \approx 2.71828$ (that the limit is exactly e is a fact calculus proves).
4. Graphs vary. Grade these features: an open circle at (−2, 3) that the graph reaches from both sides, with a closed dot at (−2, −1); a piece ending at the closed dot (1, 2) from the left and a piece starting at the open circle (1, −1) on the right; a dashed line $x = 4$ with the graph running down along it on both sides; far right the graph settling onto the x-axis; far left the graph rising without bound, with an arrow.
5. (a) It can: for example $f(x) = 2 + \frac{\sin x}{x}$ (x in radians), or a graph that waves above and below $y = 2$ with swings that shrink toward 0. (b) It cannot. (c) It can: for example $f(x) = 3$ for $x < 0$ and $f(x) = \sin\left(\frac{1}{x}\right)$ for $x > 0$.

WORKED SOLUTIONS

1. The graph of a function g is drawn below. Its domain is $x \geq -6$, the tick marks are 1 unit apart, and the dashed lines are $x = 4$ and $y = 1$.



(a) Find $\lim_{x\to -6^+} g(x)$. Why is $\lim_{x\to -6} g(x)$ not asked for?

(b) Find $\lim_{x\to -3^-} g(x)$, $\lim_{x\to -3^+} g(x)$, $\lim_{x\to -3} g(x)$ and $g(-3)$.

(c) Find $\lim_{x\to 1} g(x)$ and $g(1)$.

(d) Describe $g(x)$ as $x \to 4^-$ and as $x \to 4^+$, in limit notation. Is there a number L with $\lim_{x\to 4} g(x) = L$?

(e) Find $\lim_{x\to\infty} g(x)$.

ANSWER

(a) $\lim_{x\to -6^+} g(x) = -1$; g has no inputs to the left of −6, so only the right-hand limit can be asked. (b) 2, −2, does not exist, and $g(-3) = 2$. (c) $\lim_{x\to 1} g(x) = 2$, and $g(1)$ is undefined. (d) $\lim_{x\to 4^-} g(x) = -\infty$ and $\lim_{x\to 4^+} g(x) = -\infty$, so $\lim_{x\to 4} g(x) = -\infty$; there is no such number L, the limit does not exist. (e) $\lim_{x\to\infty} g(x) = 1$.

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SOLUTION

(a) The graph starts at the closed dot $(-6, -1)$ and runs to the right, so sliding toward -6 from the right reaches height -1: $\lim_{x \to -6^+} g(x) = -1$. The domain is $x \geq -6$, so there are no inputs on the left and a two-sided limit cannot be asked.

(b) From the left the first line rises to the closed dot $(-3, 2)$: $\lim_{x \to -3^-} g(x) = 2$. From the right the second line starts at the open circle $(-3, -2)$: $\lim_{x \to -3^+} g(x) = -2$. The two sides disagree, so $\lim_{x \to -3} g(x)$ does not exist: a jump. The closed dot is on the graph, so $g(-3) = 2$.

(c) From both sides the line heads into the open circle $(1, 2)$, so $\lim_{x \to 1} g(x) = 2$. No dot sits above 1, so $g(1)$ is undefined: a hole.

(d) From the left the curve falls along the dashed line past every height, and from the right it comes up from below every height. So $\lim_{x \to 4^-} g(x) = -\infty$ and $\lim_{x \to 4^+} g(x) = -\infty$, and since both sides go the same way we write $\lim_{x \to 4} g(x) = -\infty$. The symbol $-\infty$ is not a number: it records how the limit fails, so no number L works and the limit does not exist.

(e) Far to the right the curve settles onto the dashed line $y = 1$ from below, so $\lim_{x \to \infty} g(x) = 1$.

2. A series $a_1 + a_2 + a_3 + \cdots$ has partial sums $S_n = \dfrac{5n}{n+2}$.

(a) Find S_1, S_2, S_3, S_{98} and S_{998}.

(b) Find $\lim_{n \to \infty} S_n$, and explain why the partial sums close in on it. What is the sum of the series?

(c) Find a_1, a_2 and a_3, and show that the series is not geometric.

ANSWER

(a) $S_1 = \dfrac{5}{3}$, $S_2 = \dfrac{5}{2}$, $S_3 = 3$, $S_{98} = 4.9$, $S_{998} = 4.99$. (b) $\lim_{n \to \infty} S_n = 5$, since the gap $5 - S_n = \dfrac{10}{n+2}$ shrinks toward 0; the sum of the series is 5. (c) $a_1 = \dfrac{5}{3}$, $a_2 = \dfrac{5}{6}$, $a_3 = \dfrac{1}{2}$; the ratios $\dfrac{a_2}{a_1} = \dfrac{1}{2}$ and $\dfrac{a_3}{a_2} = \dfrac{3}{5}$ differ, so the series is not geometric.

SOLUTION

(a) $S_1 = \dfrac{5}{3}$, $S_2 = \dfrac{10}{4} = \dfrac{5}{2}$, $S_3 = \dfrac{15}{5} = 3$, $S_{98} = \dfrac{490}{100} = 4.9$ and $S_{998} = \dfrac{4{,}990}{1{,}000} = 4.99$.

(b) The partial sums rise toward 5. The gap is

$$5 - S_n = \frac{5(n+2) - 5n}{n+2} = \frac{10}{n+2},$$

which is positive and as small as you like once n is large enough (at $n = 998$ it is 0.01). So $\lim_{n \to \infty} S_n = 5$, and the sum of the series, which is defined as this limit, is 5.

(c) Each term is the change in the partial sums: $a_1 = S_1 = \dfrac{5}{3}$, $a_2 = S_2 - S_1 = \dfrac{5}{2} - \dfrac{5}{3} = \dfrac{5}{6}$ and $a_3 = S_3 - S_2 = 3 - \dfrac{5}{2} = \dfrac{1}{2}$. The ratios $\dfrac{5}{6} \div \dfrac{5}{3} = \dfrac{1}{2}$ and $\dfrac{1}{2} \div \dfrac{5}{6} = \dfrac{3}{5}$ are different, so there is no common ratio and the formula $\dfrac{a_1}{1-r}$ does not apply. The series still has a sum: the sum is the limit of the partial sums, whatever kind of series it is.

3. Let $f(x) = (1 + x)^{1/x}$, which is undefined at $x = 0$. Use a calculator.

(a) Find $f(x)$ at $x = -0.1$, -0.01, -0.001, 0.001, 0.01 and 0.1, to five decimal places.

(b) Describe how the outputs on each side behave, and estimate $\lim_{x \to 0} f(x)$ to three decimal places.

(c) The limit is a number met earlier in the course. Which number?

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ANSWER

(a) $f(-0.1) = 2.86797$, $f(-0.01) = 2.73200$, $f(-0.001) = 2.71964$, $f(0.001) = 2.71692$, $f(0.01) = 2.70481$, $f(0.1) = 2.59374$. (b) The left outputs fall and the right outputs rise toward the same number; $\lim_{x \to 0} f(x) \approx 2.718$. (c) The number $e \approx 2.71828$ (that the limit is exactly e is a fact calculus proves).

SOLUTION

(a) For instance $f(-0.1) = 0.9^{-10} = \dfrac{1}{0.9^{10}} \approx 2.86797$ and $f(0.1) = 1.1^{10} \approx 2.59374$.

x **(left)**	$f(x)$	x **(right)**	$f(x)$
-0.1	2.86797	0.1	2.59374
-0.01	2.73200	0.01	2.70481
-0.001	2.71964	0.001	2.71692

(b) As the inputs close in on 0, the left outputs fall and the right outputs rise. The gap between the two sides is 0.27423, then 0.02719, then 0.00272, shrinking about tenfold each row, so both sides close in on one number between 2.71692 and 2.71964, near 2.718. So $\lim_{x \to 0} f(x) \approx 2.718$.

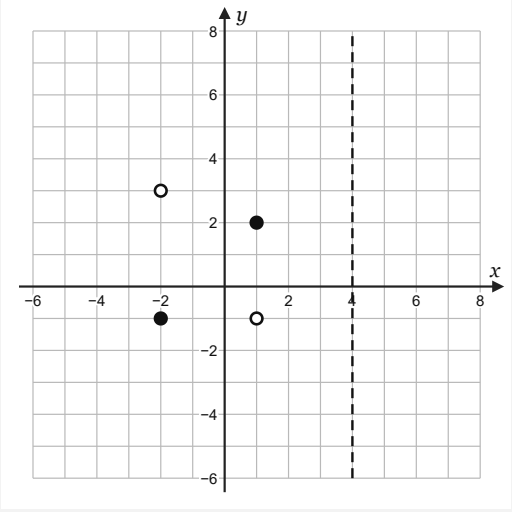
(c) The estimate matches $e \approx 2.71828$, the base of the natural exponential function. The table only estimates the limit; that it equals e exactly is a fact calculus proves.

4. Sketch the graph of one function f that has all of these properties. Draw each closed dot and open circle, draw each asymptote dashed, and use arrows where the graph goes on.

- $\lim_{x \to -2} f(x) = 3$, and $f(-2) = -1$
- $\lim_{x \to 1^-} f(x) = 2$, $\lim_{x \to 1^+} f(x) = -1$, and $f(1) = 2$
- $\lim_{x \to 4} f(x) = -\infty$
- $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = \infty$

ANSWER

Graphs vary. Grade these features: an open circle at (−2, 3) that the graph reaches from both sides, with a closed dot at (−2, −1); a piece ending at the closed dot (1, 2) from the left and a piece starting at the open circle (1, −1) on the right; a dashed line $x = 4$ with the graph running down along it on both sides; far right the graph settling onto the x-axis; far left the graph rising without bound, with an arrow.



SOLUTION

Work through the list one input at a time.

At $x = -2$ the two-sided limit is 3, so the graph heads into the height 3 from both sides; the value is −1, so that point is an open circle at (−2, 3) and a closed dot sits at (−2, −1).

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At $x = 1$ the left side heads to 2 and the value is 2, so the left piece ends at a closed dot (1, 2). The right side heads to −1, so the right piece starts at an open circle (1, −1): a jump.

At $x = 4$ both sides fall past every height, so draw the dashed line $x = 4$ with the graph running down along it on both sides; there is no point above 4.

Far to the right the graph settles onto the line $y = 0$ (the x-axis), and far to the left it rises without bound, so it ends in an arrow pointing up and to the left.

One graph that works: a curve falling from the upper left through the open circle (−2, 3) to the closed dot (1, 2); a curve from the open circle (1, −1) falling down along $x = 4$; and on the right of $x = 4$ a curve coming up from below and rising toward the x-axis. Any graph with the listed features is correct.

5. Decide whether each situation can happen. If it can, give an example (a formula or a described graph). If it cannot, explain why.

(a) The graph of f crosses the line $y = 2$ infinitely many times, and $\lim_{x \to \infty} f(x) = 2$.

(b) $\lim_{x \to 1} f(x) = 5$ and $\lim_{x \to 1^+} f(x) = 4$.

(c) $\lim_{x \to 0^-} f(x) = 3$, while $\lim_{x \to 0^+} f(x)$ does not exist.

ANSWER

(a) It can: for example $f(x) = 2 + \dfrac{\sin x}{x}$ (x in radians), or a graph that waves above and below $y = 2$ with swings that shrink toward 0. (b) It cannot. (c) It can: for example $f(x) = 3$ for $x < 0$ and $f(x) = \sin\left(\dfrac{1}{x}\right)$ for $x > 0$.

SOLUTION

(a) For $f(x) = 2 + \dfrac{\sin x}{x}$, $f(x) = 2$ whenever $\sin x = 0$, at $x = \pi, 2\pi, 3\pi, \ldots$, so the graph crosses $y = 2$ infinitely many times. For $x > 0$ the distance from $f(x)$ to 2 is $\left|\dfrac{\sin x}{x}\right| \le \dfrac{1}{x}$, which is as small as you like once x is large enough. So $\lim_{x \to \infty} f(x) = 2$. Only the far-out behavior counts, not whether the graph touches the line. A described graph is also correct: a wave about $y = 2$ that crosses it again and again while its swings shrink toward 0.

(b) A two-sided limit exists exactly when both one-sided limits exist and are equal, and then it is their common value. If $\lim_{x \to 1} f(x) = 5$, then $\lim_{x \to 1^+} f(x)$ must also be 5, so it cannot be 4.

(c) For $f(x) = 3$ when $x < 0$, every output just left of 0 is 3, so $\lim_{x \to 0^-} f(x) = 3$. For $x > 0$, $\sin\left(\dfrac{1}{x}\right)$ keeps swinging between 1 and −1 however close x comes to 0, so $\lim_{x \to 0^+} f(x)$ does not exist. (Then $\lim_{x \to 0} f(x)$ does not exist either.)

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