Calculus 1 Unit 1 worksheet: The limit laws (answer key)

ANSWERS AT A GLANCE

1. (a) 10 (b) −125 (c) $\frac{12}{5}$ (d) 2 (e) $-\pi$ (f) 3
2. $\frac{31}{11}$
3. $\frac{7\sqrt{2}}{2}$
4. (a) $f(x) = \frac{1}{2}\Big[\big(f(x)+g(x)\big)+\big(f(x)-g(x)\big)\Big]$, so $\lim_{x\to -3} f(x) = 4$. (b) $\lim_{x\to -3} g(x) = -5$ and $\lim_{x\to -3} f(x)\,g(x) = -20$.
5. (a) The law applies: the limit is $\frac{1}{2}$. (b) The law does not apply (bottom limit 0, top limit 5); the limit does not exist. (c) The law does not apply: the form is $\frac{0}{0}$, which by itself says nothing about the limit.
6. (a) The first step: the product law needs both limits to exist, and $\lim_{x\to 0}\left(\frac{3}{x^2}+1\right)$ does not exist. (b) 3 (c) Yes; the limit is 4.

WORKED SOLUTIONS

1. Find each limit. Name the law or result that lets you find it (for example the product law, the power law, or direct substitution for a rational function whose bottom is not 0).

(a) $\lim_{x\to -2}(x^3+3x^2-x+4)$

(b) $\lim_{x\to 3}(4-x^2)^3$

(c) $\lim_{x\to 4}\dfrac{x^2-4}{x+1}$

(d) $\lim_{x\to -3}\left(x+\sqrt{x^2+16}\right)$

(e) $\lim_{x\to \pi} x\cos x$

(f) $\lim_{x\to -1}\dfrac{3^{x^2+x+2}}{x+4}$

ANSWER

(a) 10 (b) −125 (c) $\frac{12}{5}$ (d) 2 (e) $-\pi$ (f) 3

SOLUTION

(a) A polynomial, so its limit is its value: $(-2)^3+3(-2)^2-(-2)+4 = -8+12+2+4 = 10$.

(b) The inside is a polynomial with limit $4-3^2=-5$, so by the power law the limit is $(-5)^3=-125$.

(c) The top tends to 16 − 4 = 12 and the bottom to 4 + 1 = 5 ≠ 0, so the quotient law (direct substitution for a rational function) gives $\frac{12}{5}$.

(d) The inside of the root tends to 9 + 16 = 25 > 0, so the root law gives $\sqrt{x^2+16}\to 5$. With $\lim_{x\to -3} x = -3$, the sum law gives −3 + 5 = 2.

(e) $\lim_{x\to\pi} x = \pi$ and $\cos x$ is continuous, so $\lim_{x\to\pi}\cos x = \cos\pi = -1$. By the product law the limit is $\pi\cdot(-1) = -\pi$.

(f) The exponent is a polynomial with limit 1 − 1 + 2 = 2, and 3^u is continuous everywhere, so by the limit of a composition the top tends to $3^2=9$. The bottom tends to −1 + 4 = 3 ≠ 0, so the quotient law gives $\frac{9}{3}=3$.

2. Suppose $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = -5$. Find the limit below, naming each law you use, and show that the quotient law is allowed.

$$\lim_{x\to 3}\frac{4f(x)-x\,g(x)}{\big(f(x)\big)^2+g(x)}$$

ANSWER

$\frac{31}{11}$

SOLUTION

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Both given limits exist, so the laws apply. For the top: the constant multiple law gives $4f(x) \to 4 \cdot 4$, and the product law with $\lim_{x \to 3} x = 3$ gives $x\,g(x) \to 3 \cdot (-5)$. By the difference law the top tends to

$$4 \cdot 4 - 3 \cdot (-5) = 31$$

For the bottom: the power law gives $\big(f(x)\big)^2 \to 4^2$, and the sum law adds $\lim g(x)$:

$$4^2 - 5 = 11$$

The bottom limit is $11 \neq 0$, so the quotient law gives

$$\lim_{x \to 3} \frac{4f(x) - x\,g(x)}{\big(f(x)\big)^2 + g(x)} = \frac{31}{11}$$

In lowest terms the limit is $\dfrac{31}{11}$.

3. Find $\lim_{x \to 5} \sqrt{\dfrac{2x^2 - 1}{x - 3}}$ exactly. Say which law or result each step uses, and why each one is allowed.

ANSWER

$\dfrac{7\sqrt{2}}{2}$

SOLUTION

This is a composition: the square root is fed the rational function $\dfrac{2x^2 - 1}{x - 3}$.

Inner limit. The top and bottom are polynomials: $2x^2 - 1 \to 2 \cdot 25 - 1 = 49$ and $x - 3 \to 5 - 3 = 2$. The bottom limit is $2 \neq 0$, so the quotient law gives

$$\lim_{x \to 5} \frac{2x^2 - 1}{x - 3} = \frac{49}{2}$$

Outer function. The inner limit $\dfrac{49}{2}$ is positive, so the square root is continuous there and the root law applies:

$$\lim_{x \to 5} \sqrt{\frac{2x^2 - 1}{x - 3}} = \sqrt{\frac{49}{2}} = \frac{7}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$$

4. Suppose $\lim_{x \to -3} \big[f(x) + g(x)\big] = -1$ and $\lim_{x \to -3} \big[f(x) - g(x)\big] = 9$. Nothing else is known about f and g.
(a) Write $f(x)$ using only $f(x) + g(x)$ and $f(x) - g(x)$. Use the laws to explain why $\lim_{x \to -3} f(x)$ exists, and find it.

(b) Find $\lim_{x \to -3} g(x)$ and $\lim_{x \to -3} f(x)\,g(x)$.

ANSWER

(a) $f(x) = \dfrac{1}{2}\Big[\big(f(x) + g(x)\big) + \big(f(x) - g(x)\big)\Big]$, so $\lim_{x \to -3} f(x) = 4$. (b) $\lim_{x \to -3} g(x) = -5$ and $\lim_{x \to -3} f(x)\,g(x) = -20$.

SOLUTION

(a) Adding the two expressions gives $2f(x)$, so

$$f(x) = \frac{1}{2}\Big[\big(f(x) + g(x)\big) + \big(f(x) - g(x)\big)\Big]$$

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Both limits inside the brackets exist, so the sum law and then the constant multiple law apply:

$$\lim_{x\to -3} f(x) = \frac{1}{2}\big(-1+9\big) = 4$$

This is the laws used forward, on two limits that are known to exist; it is not splitting a limit into pieces that might not exist.

(b) Subtracting instead gives $g(x) = \dfrac{1}{2}\Big[\big(f(x)+g(x)\big) - \big(f(x)-g(x)\big)\Big]$, so by the difference and constant multiple laws

$$\lim_{x\to -3} g(x) = \frac{1}{2}\big(-1-9\big) = -5$$

Now both $\lim f(x)$ and $\lim g(x)$ exist, so the product law gives $\lim_{x\to -3} f(x)\,g(x) = 4\cdot(-5) = -20$.

5. For each limit, find the limits of the top and of the bottom, and decide whether the quotient law applies. If it does, give the limit. If it does not, say why, and say whether you can already tell that the limit does not exist. (A $\dfrac{0}{0}$ form is finished by the algebra of the next lesson; here only name it.)

(a) $\lim_{x\to 0} \dfrac{\cos x + 1}{x^2+4}$ (b) $\lim_{x\to 2} \dfrac{x^2+1}{\sqrt{x+2}-2}$ (c) $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2-4}$

ANSWER

(a) The law applies: the limit is $\dfrac{1}{2}$. (b) The law does not apply (bottom limit 0, top limit 5); the limit does not exist. (c) The law does not apply: the form is $\dfrac{0}{0}$, which by itself says nothing about the limit.

SOLUTION

(a) $\cos x$ is continuous, so the top tends to $\cos 0 + 1 = 2$. The bottom tends to $0 + 4 = 4 \neq 0$. The quotient law gives $\dfrac{2}{4} = \dfrac{1}{2}$.

(b) Top: $4 + 1 = 5$. Bottom: $x+2 \to 4 > 0$, so the root law gives $\sqrt{x+2} \to 2$ and the difference law gives $2 - 2 = 0$. The quotient law needs a nonzero bottom limit, so it does not apply. Near 2 the outputs are about 5 divided by numbers closer and closer to 0, so they grow without bound in size, and the limit does not exist.

(c) Top: $4 - 2 - 2 = 0$. Bottom: $4 - 4 = 0$. The quotient law does not apply. Both limits are 0, and the form $\dfrac{0}{0}$ by itself does not decide whether the limit exists or what it is; that takes the factoring and rationalizing of the next lesson.

6. Yuna finds a limit this way:

$$\begin{aligned}\lim_{x\to 0} x^2\left(\frac{3}{x^2}+1\right) &= \lim_{x\to 0} x^2 \cdot \lim_{x\to 0}\left(\frac{3}{x^2}+1\right)\\ &= 0\cdot \lim_{x\to 0}\left(\frac{3}{x^2}+1\right) = 0\end{aligned}$$

(a) Which step uses a law whose hypothesis fails? Explain.

(b) Find the limit correctly.

(c) Is the same first step allowed for $\lim_{x\to 1} x^2\left(\dfrac{3}{x^2}+1\right)$? Find that limit.

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ANSWER

(a) The first step: the product law needs both limits to exist, and $\lim_{x \to 0} \left(\dfrac{3}{x^2} + 1 \right)$ does not exist. (b) 3 (c) Yes; the limit is 4.

SOLUTION

(a) The first equals sign uses the product law. That law holds only when both factors have limits that are numbers. As $x \to 0$, $\dfrac{3}{x^2}$ is 3 divided by small positive numbers, so it grows without bound, and $\lim_{x \to 0} \left(\dfrac{3}{x^2} + 1 \right)$ does not exist. So "$0 \cdot \lim(\ldots)$" is not a product of two numbers, and the 0 at the end is not justified.

(b) The limit looks only at $x \neq 0$, and there

$$x^2 \left(\frac{3}{x^2} + 1 \right) = 3 + x^2$$

a polynomial, so $\lim_{x \to 0} x^2 \left(\dfrac{3}{x^2} + 1 \right) = \lim_{x \to 0} (3 + x^2) = 3$.

(c) Yes. Now $\lim_{x \to 1} x^2 = 1$, and by the quotient law (bottom limit $1 \neq 0$) and the sum law $\lim_{x \to 1} \left(\dfrac{3}{x^2} + 1 \right) = 3 + 1 = 4$. Both limits exist, so the product law gives $1 \cdot 4 = 4$.

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