Calculus 1 Unit 1 worksheet: One sided limits and when a limit exists (answer key)

ANSWERS AT A GLANCE

1. $\lim_{t\to 4^-} C(t) = 92$ dollars, $\lim_{t\to 4^+} C(t) = 92$ dollars, $C(4) = 92$ dollars; the two-sided limit exists, so the price has no jump there. Rentals a little under 4 hours cost close to 92 dollars, and rentals a little over cost close to 92 dollars.
2. (a) -4; from the right 2; the limit does not exist. (b) 3; from the right 0; the limit does not exist. (c) 0; from the left 0; the limit exists and equals 0.
3. $\lim_{x\to 4^-} f(x) = -8$ and $\lim_{x\to 4^+} f(x) = 8$, so $\lim_{x\to 4} f(x)$ does not exist; $\lim_{x\to -4} f(x) = 0$.
4. (a) $x \le -3$ or $x \ge 3$. (b) At 3: only $\lim_{x\to 3^+} f(x) = 0$. At -3: only $\lim_{x\to -3^-} f(x) = 0$. At 5: $\lim_{x\to 5} f(x) = 4$. (c) $\lim_{x\to 3^-} g(x) = 0$ and $\lim_{x\to 3^+} g(x) = 0$, so $\lim_{x\to 3} g(x) = 0$. (d) 12.
5. (a) $\lim_{h\to 20^-} D(h) = 4$, $\lim_{h\to 20^+} D(h) = 5$, $D(20) = 5$; the limit does not exist. (b) 6. (c) $a =$ 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44 (the eleven multiples of 4 between 0 and 48). (d) At 0 only $\lim_{h\to 0^+} D(h) = 0$; at 48 only $\lim_{h\to 48^-} D(h) = 11$.
6. (a) Answers vary. One example:

$$f(x) = \begin{cases} \dfrac{8-2x}{3}, & x < 4,\ x \neq 1 \\ 5, & x = 1 \\ 1, & x \ge 4 \end{cases}$$

(b) No. If both one-sided limits at 4 were 0, the criterion would give $\lim_{x\to 4} f(x) = 0$, so the limit would exist.

WORKED SOLUTIONS

1. Diego rents a kayak. For a rental of t hours the shop charges $C(t) = 12 + 20t$ dollars when $0 < t \le 4$, and $C(t) = 24 + 17t$ dollars when $t > 4$. Find $\lim_{t\to 4^-} C(t)$, $\lim_{t\to 4^+} C(t)$ and $C(4)$, and decide whether $\lim_{t\to 4} C(t)$ exists. What do your answers say about rentals a few minutes shorter and a few minutes longer than 4 hours?

ANSWER

$\lim_{t\to 4^-} C(t) = 92$ dollars, $\lim_{t\to 4^+} C(t) = 92$ dollars, $C(4) = 92$ dollars; the two-sided limit exists, so the price has no jump there. Rentals a little under 4 hours cost close to 92 dollars, and rentals a little over cost close to 92 dollars.

SOLUTION

For t just less than 4 the first rule is in force, a polynomial, so by locality

$$\lim_{t\to 4^-} C(t) = 12 + 20 \cdot 4 = 12 + 80 = 92$$

For t just greater than 4 the second rule is in force, so

$$\lim_{t\to 4^+} C(t) = 24 + 17 \cdot 4 = 24 + 68 = 92$$

The condition $0 < t \le 4$ includes 4, so $C(4) = 92$. The one-sided limits are 92 and 92, so by the criterion the two-sided limit exists, so the price has no jump there. In dollars: rentals just under 4 hours cost close to 92 dollars and rentals just over cost close to 92 dollars.

2. Find each one-sided limit from the rule that governs that side. Then find the limit from the other side, and decide whether the two-sided limit exists.

(a) $\lim_{x\to 0^-} \dfrac{|3x| - x}{x}$

(b) $\lim_{x\to 2^-} \left(x^2 - \lfloor x \rfloor^2\right)$

(c) $\lim_{x\to 2^+} \dfrac{x^2 - 4x + 4}{|x - 2|}$

ANSWER

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(a) -4; from the right 2; the limit does not exist. (b) 3; from the right 0; the limit does not exist. (c) 0; from the left 0; the limit exists and equals 0.

SOLUTION

(a) For $x < 0$, $|3x| = -3x$, so the quotient is $\dfrac{-3x - x}{x} = -4$ for every such x: the left-hand limit is -4. For $x > 0$ it is $\dfrac{3x - x}{x} = 2$. The sides differ, so the limit does not exist.

(b) For $1 \le x < 2$, $\lfloor x \rfloor = 1$, so the rule is $x^2 - 1$, which heads to 3. For $2 \le x < 3$ the rule is $x^2 - 4$, which heads to 0. The limit does not exist.

(c) The top is $(x-2)^2 = |x-2|^2$, so for $x \neq 2$ the quotient equals $|x-2|$, which is $x - 2$ on the right and $2 - x$ on the left. Both head to 0, so the limit exists and equals 0. The square on top has no sign change at 2, so unlike (a) there is no jump.

3. Let $f(x) = \dfrac{|x^2 - 16|}{x - 4}$. Find $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$, and decide whether $\lim_{x \to 4} f(x)$ exists. Then find $\lim_{x \to -4} f(x)$.

ANSWER

$\lim_{x \to 4^-} f(x) = -8$ and $\lim_{x \to 4^+} f(x) = 8$, so $\lim_{x \to 4} f(x)$ does not exist; $\lim_{x \to -4} f(x) = 0$.

SOLUTION

The quantity inside the absolute value factors as $(x-4)(x+4)$, so $|x^2 - 16| = |x - 4| \cdot |x + 4|$.

Near 4, $x + 4$ is positive, so $|x + 4| = x + 4$. For $x < 4$, $|x - 4| = -(x - 4)$, and

$$f(x) = \frac{-(x-4)(x+4)}{x - 4} = -(x + 4)$$

which heads to -8. For $x > 4$, $|x - 4| = x - 4$ and $f(x) = x + 4$, which heads to 8. The one-sided limits differ, so $\lim_{x \to 4} f(x)$ does not exist.

Near -4, $x - 4$ is negative, so $|x - 4| = -(x - 4)$ and $f(x) = -|x + 4|$ on both sides of -4. Both one-sided limits are 0, so $\lim_{x \to -4} f(x) = 0$. The absolute value makes a jump only where the factor it shares with the bottom changes sign.

4. Let $f(x) = \sqrt{x^2 - 9}$.

(a) Find the domain of f.

(b) For each of $a = 3$, $a = -3$ and $a = 5$, say which limit of f at a is meaningful (left, right or two-sided) and find it.

(c) Let $g(x) = \sqrt{x^2 - 9}$ for $x \ge 3$ and $g(x) = 6 - 2x$ for $x < 3$. Find both one-sided limits of g at 3, and decide whether $\lim_{x \to 3} g(x)$ exists.

(d) Find $\lim_{x \to -3} g(x)$.

ANSWER

(a) $x \le -3$ or $x \ge 3$. (b) At 3: only $\lim_{x \to 3^+} f(x) = 0$. At -3: only $\lim_{x \to -3^-} f(x) = 0$. At 5: $\lim_{x \to 5} f(x) = 4$. (c) $\lim_{x \to 3^-} g(x) = 0$ and $\lim_{x \to 3^+} g(x) = 0$, so $\lim_{x \to 3} g(x) = 0$. (d) 12.

SOLUTION

(a) The root needs $x^2 - 9 \ge 0$, that is $x^2 \ge 9$: $x \le -3$ or $x \ge 3$.

(b) The inputs strictly between -3 and 3 are missing. At 3 there are inputs only to the right, so only the right-hand limit is meaningful. For $3 < x < 4$, $x^2 - 9 = (x-3)(x+3) < 7(x-3)$, so for a tolerance $t > 0$ every x with $3 < x < 3 + \dfrac{t^2}{7}$ (and $x < 4$) gives $\sqrt{x^2 - 9} < t$. Hence $\lim_{x \to 3^+} f(x) = 0$. By the same

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argument on the other side, at -3 only $\lim_{x\to -3^-} f(x) = 0$ is meaningful. At 5, inside the domain, both sides exist and $\lim_{x\to 5} f(x) = \sqrt{25-9} = 4$.

(c) From the left the rule is the polynomial $6 - 2x$, so $\lim_{x\to 3^-} g(x) = 6 - 6 = 0$. From the right the rule is $\sqrt{x^2 - 9}$, whose right-hand limit at 3 is 0 by (b). The sides agree, so $\lim_{x\to 3} g(x) = 0$. The function g has inputs on both sides of 3, though one of its rules does not.

(d) Every input near -3 is less than 3, so near -3 the rule of g is $6 - 2x$ on both sides, and $\lim_{x\to -3} g(x) = 6 + 6 = 12$. The root rule, which stops at -3 for f, is not used there by g at all.

5. Mei rides an elevator in an office tower whose floors are 4 meters apart; the lobby is floor 0. When the floor of the car is h meters above the lobby floor, the display shows the floor number

$$D(h) = \left\lfloor \frac{h}{4} \right\rfloor, \qquad 0 \le h \le 48$$

(a) Find $\lim_{h\to 20^-} D(h)$, $\lim_{h\to 20^+} D(h)$ and $D(20)$. Does $\lim_{h\to 20} D(h)$ exist?

(b) Find $\lim_{h\to 27} D(h)$.

(c) Find every height a with $0 < a < 48$ at which $\lim_{h\to a} D(h)$ does not exist.

(d) At $h = 0$ and at $h = 48$, which one-sided limit is meaningful, and what is it?

ANSWER

(a) $\lim_{h\to 20^-} D(h) = 4$, $\lim_{h\to 20^+} D(h) = 5$, $D(20) = 5$; the limit does not exist. (b) 6. (c) $a = 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44$ (the eleven multiples of 4 between 0 and 48). (d) At 0 only $\lim_{h\to 0^+} D(h) = 0$; at 48 only $\lim_{h\to 48^-} D(h) = 11$.

SOLUTION

(a) For $16 \le h < 20$, $4 \le \dfrac{h}{4} < 5$, so $D(h) = 4$ and $\lim_{h\to 20^-} D(h) = 4$. For $20 \le h < 24$, $D(h) = 5$, so $\lim_{h\to 20^+} D(h) = 5$, and $D(20) = 5$. The sides differ, so $\lim_{h\to 20} D(h)$ does not exist: the display changes as the car passes 20 meters.

(b) Every h between 24 and 28 gives $6 \le \dfrac{h}{4} < 7$, so $D(h) = 6$ on a whole window around 27 and $\lim_{h\to 27} D(h) = 6$.

(c) At a height a that is not a multiple of 4, a window around a lies between two consecutive multiples of 4, where D is constant, so the limit exists. At $a = 4j$ the left side gives $j - 1$ and the right side gives j, so the limit fails. Between 0 and 48 that is $a = 4, 8, \ldots, 44$, eleven heights.

(d) D is defined only for $0 \le h \le 48$. At 0 there are no inputs to the left, so only $\lim_{h\to 0^+} D(h) = 0$ is meaningful ($D(h) = 0$ for $0 \le h < 4$). At 48 there are no inputs to the right, so only $\lim_{h\to 48^-} D(h)$ is meaningful; for $44 \le h < 48$, $D(h) = 11$, so it is 11, though $D(48) = 12$.

6. (a) Write a formula for a function f, defined for every real x, with all of these properties: $\lim_{x\to 1} f(x) = 2$; $f(1) = 5$; $\lim_{x\to 4^-} f(x) = 0$; and $\lim_{x\to 4} f(x)$ does not exist. Check each property from your formula.

(b) Could a function with these four properties also have $\lim_{x\to 4^+} f(x) = 0$? Explain using the criterion.

ANSWER

(a) Answers vary. One example:

$$f(x) = \begin{cases} \dfrac{8 - 2x}{3}, & x < 4,\ x \ne 1 \\ 5, & x = 1 \\ 1, & x \ge 4 \end{cases}$$

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(b) No. If both one-sided limits at 4 were 0, the criterion would give $\lim_{x \to 4} f(x) = 0$, so the limit would exist.

SOLUTION

(a) In the example, near 1 the rule on both sides is $\dfrac{8 - 2x}{3}$, a polynomial, so $\lim_{x \to 1} f(x) = \dfrac{8 - 2}{3} = 2$; the separate line gives $f(1) = 5$, which the limit never consults. From the left of 4 the rule is again $\dfrac{8 - 2x}{3}$, so $\lim_{x \to 4^-} f(x) = \dfrac{8 - 8}{3} = 0$. From the right the rule is the constant 1, so $\lim_{x \to 4^+} f(x) = 1 \neq 0$, and by the criterion $\lim_{x \to 4} f(x)$ does not exist. Any formula is correct if it passes these four checks; for instance the right side could also run off to ∞, as $\dfrac{1}{x - 4}$ does.

(b) The left-hand limit at 4 is 0. If the right-hand limit were 0 too, both one-sided limits would exist and agree, and the criterion says the two-sided limit then exists and equals 0. That contradicts the fourth property, so the right-hand limit must be a different number or fail to exist.

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