Calculus 1 Unit 1 worksheet: Infinite limits and vertical asymptotes (answer key)

ANSWERS AT A GLANCE

1. (a) $-\infty$ (b) ∞ (c) $-\infty$ (d) does not exist ($-\infty$ from the left, ∞ from the right) (e) $-\infty$
2. $\lim_{x\to -1} f(x) = \infty$; $\lim_{x\to -4^-} f(x) = \infty$ and $\lim_{x\to -4^+} f(x) = -\infty$
3. (a) $-\infty$ from the left and ∞ from the right. (b) ∞. For $x > 2$, $4 - x^2 < 0$, so the function is not defined to the right of 2; one side is enough, so $x = 2$ is a vertical asymptote.
4. (a) $-\infty$ from the left and ∞ from the right. (b) ∞. (c) Yes, of both.
5. (a) $\delta = \dfrac{8}{M^3}$ (b) $\delta = \dfrac{1}{125{,}000}$ (c) $-\infty$
6. (a) ∞ (b) $-\infty$ (c) $-\infty$ (d) no conclusion (e) 0 (f) no conclusion

WORKED SOLUTIONS

1. Find each limit. Write ∞ or $-\infty$ when the outputs grow without bound, and write "does not exist" when the two sides disagree. Show the sign of the top and of each factor of the bottom.

(a) $\displaystyle\lim_{x\to 2^+} \frac{3x+1}{2-x}$

(b) $\displaystyle\lim_{x\to 5^-} \frac{x-8}{x^2-25}$

(c) $\displaystyle\lim_{x\to 3} \frac{6-x^2}{x^2-6x+9}$

(d) $\displaystyle\lim_{x\to 1} \frac{2x+3}{x^3-1}$

(e) $\displaystyle\lim_{x\to -2^+} \frac{1-x}{x^2+x-2}$

ANSWER

(a) $-\infty$ (b) ∞ (c) $-\infty$ (d) does not exist ($-\infty$ from the left, ∞ from the right) (e) $-\infty$

SOLUTION

In every part the top tends to a nonzero number and the bottom tends to 0, so the size grows without bound and only the sign is left to decide.

(a) The top tends to 7 > 0. For $x > 2$, $2 - x < 0$. Positive over negative: $-\infty$.

(b) The top tends to −3. The bottom is $(x-5)(x+5)$; for x just below 5, $x - 5 < 0$ and $x + 5 > 0$, so the bottom is negative. Negative over negative: ∞.

(c) The top tends to 6 − 9 = −3. The bottom is $(x-3)^2$, positive on both sides of 3. Negative over positive on both sides: $-\infty$.

(d) The top tends to 5. The bottom is $(x-1)(x^2+x+1)$, and $x^2 + x + 1 \to 3 > 0$, so the bottom has the sign of $x - 1$: negative on the left, positive on the right. So $\lim_{x\to 1^-} = -\infty$ and $\lim_{x\to 1^+} = \infty$; the sides go opposite ways, so the limit does not exist.

(e) The top tends to 1 − (−2) = 3. The bottom is $(x+2)(x-1)$; for x just above −2, $x + 2 > 0$ and $x - 1 \to -3 < 0$, so the bottom is negative. Positive over negative: $-\infty$.

2. Let $f(x) = \dfrac{x+2}{(x+1)^2(x+4)}$. Find $\lim_{x\to -1} f(x)$, and find the two one-sided limits of f at $x = -4$. Explain the signs.

ANSWER

$\lim_{x\to -1} f(x) = \infty$; $\lim_{x\to -4^-} f(x) = \infty$ and $\lim_{x\to -4^+} f(x) = -\infty$

SOLUTION

Near $x = -1$: the top tends to −1 + 2 = 1 and $x + 4$ tends to −1 + 4 = 3, both nonzero, while $(x+1)^2$ tends to 0 and is positive on both sides. So on both sides $f(x)$ has the sign of $\dfrac{1}{3}$, and its size grows without bound: $\lim_{x\to -1} f(x) = \infty$, from both sides.

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Near $x = -4$: the top tends to −2 and $(x+1)^2$ tends to 9 > 0, while $x + 4$ tends to 0, negative on the left of −4 and positive on the right. On the left $f(x)$ has the sign opposite to −2, and on the right the same sign as −2. So $\lim_{x\to -4^-} f(x) = \infty$ and $\lim_{x\to -4^+} f(x) = -\infty$. The sides go opposite ways, so $\lim_{x\to -4} f(x)$ does not exist, while the squared factor gave one answer for both sides at −1.

3. Find each limit, and say which result you use.

(a) $\lim\limits_{x\to 0^-} \dfrac{2-\cos x}{\sin x}$ and $\lim\limits_{x\to 0^+} \dfrac{2-\cos x}{\sin x}$

(b) $\lim\limits_{x\to 2^-} \dfrac{x}{\sqrt{4-x^2}}$. Why is no limit from the right asked for in (b)? Is the line $x = 2$ a vertical asymptote of the graph?

ANSWER

(a) $-\infty$ from the left and ∞ from the right. (b) ∞. For $x > 2$, $4 - x^2 < 0$, so the function is not defined to the right of 2; one side is enough, so $x = 2$ is a vertical asymptote.

SOLUTION

(a) As $x \to 0$, the top $2 - \cos x \to 2 - 1 = 1 \neq 0$ and $\sin x \to 0$: nonzero over zero. For x just below 0 (in $(-\pi, 0)$), $\sin x < 0$, and for x just above 0, $\sin x > 0$. Positive over negative gives $\lim_{x\to 0^-} \dfrac{2-\cos x}{\sin x} = -\infty$; positive over positive gives $\lim_{x\to 0^+} \dfrac{2-\cos x}{\sin x} = \infty$.

(b) As $x \to 2^-$, the top tends to 2 and $\sqrt{4-x^2} \to 0$ through positive values (a square root of a positive number). Nonzero over zero with the same signs: $\lim_{x\to 2^-} \dfrac{x}{\sqrt{4-x^2}} = \infty$.

For $x > 2$, $4 - x^2 < 0$ and the square root is not defined, so there is no right side to approach from. The definition of a vertical asymptote needs only one of the four one-sided limits to be infinite, so $x = 2$ is a vertical asymptote.

4. (a) Write $\dfrac{\sin 3x}{x^2}$ as $\dfrac{\sin 3x}{3x} \cdot \dfrac{3}{x}$, and use it to find $\lim\limits_{x\to 0^-} \dfrac{\sin 3x}{x^2}$ and $\lim\limits_{x\to 0^+} \dfrac{\sin 3x}{x^2}$. Name the results you use.

(b) Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x^3}$.

(c) Is the line $x = 0$ a vertical asymptote of either graph?

ANSWER

(a) $-\infty$ from the left and ∞ from the right. (b) ∞. (c) Yes, of both.

SOLUTION

(a) The top and bottom both tend to 0, so nonzero over zero does not apply directly. Split the quotient: $\dfrac{\sin 3x}{3x} \to 1$ by the limit of $\dfrac{\sin u}{u}$ with $u = 3x$. As $x \to 0^+$, $\dfrac{3}{x} \to \infty$ (positive over positive), and as $x \to 0^-$, $\dfrac{3}{x} \to -\infty$. By the product result, a factor tending to ∞ times a factor tending to 1 > 0 tends to ∞, and with $-\infty$ it tends to $-\infty$. So the left limit is $-\infty$ and the right limit is ∞.

(b) Write $\dfrac{\sin 3x}{x^3} = \dfrac{\sin 3x}{3x} \cdot \dfrac{3}{x^2}$. The first factor tends to 1, and $\dfrac{3}{x^2} \to \infty$ from both sides, since $x^2 > 0$. By the product result the limit is ∞ from both sides, so $\lim_{x\to 0} \dfrac{\sin 3x}{x^3} = \infty$.

(c) Yes. Each function has an infinite one-sided limit at 0, and one is enough.

5. Let $g(x) = \dfrac{2}{\sqrt[3]{x-1}}$.

(a) Find a δ in terms of M that proves $\lim\limits_{x\to 1^+} g(x) = \infty$ from the definition.

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(b) Give the δ your answer to (a) gives for $M = 100$, as a fraction.

(c) Find $\lim_{x \to 1^-} g(x)$, and show that the same δ answers the bound M there.

ANSWER

(a) $\delta = \dfrac{8}{M^3}$ (b) $\delta = \dfrac{1}{125{,}000}$ (c) $-\infty$

SOLUTION

(a) Let $M > 0$. For $x > 1$ the cube root $\sqrt[3]{x-1}$ is positive, so

$$\begin{aligned} \frac{2}{\sqrt[3]{x-1}} > M &\iff \sqrt[3]{x-1} < \frac{2}{M} \\ &\iff x - 1 < \frac{8}{M^3}, \end{aligned}$$

using that cubing keeps the order of positive numbers. So $\delta = \dfrac{8}{M^3}$ works: whenever $1 < x < 1 + \dfrac{8}{M^3}$, $g(x) > M$. Since every M is answered, $\lim_{x \to 1^+} g(x) = \infty$.

(b) $\delta = \dfrac{8}{100^3} = \dfrac{8}{1{,}000{,}000} = \dfrac{1}{125{,}000}$.

(c) For $x < 1$ the cube root is negative and tends to 0, so $g(x) \to -\infty$. If $1 - \dfrac{8}{M^3} < x < 1$, then $-\dfrac{8}{M^3} < x - 1 < 0$, so $-\dfrac{2}{M} < \sqrt[3]{x-1} < 0$, and dividing 2 by a negative number of size less than $\dfrac{2}{M}$ gives $g(x) < -M$. So $\lim_{x \to 1^-} g(x) = -\infty$ with the same δ.

6. All that is known about three functions near 4 is this: $\lim_{x \to 4} f(x) = \infty$, $\lim_{x \to 4} g(x) = -2$, and $\lim_{x \to 4} h(x) = 0$ with $h(x) > 0$ for every $x \neq 4$ near 4. Find each limit as $x \to 4$, or write "no conclusion" if the facts given do not decide it. Name the result you use.

(a) $f(x) + g(x)$ (b) $f(x)\, g(x)$ (c) $\dfrac{g(x)}{h(x)}$

(d) $f(x)\, h(x)$ (e) $\dfrac{g(x)}{f(x)}$ (f) $f(x) - \dfrac{1}{h(x)}$

ANSWER

(a) ∞ (b) $-\infty$ (c) $-\infty$ (d) no conclusion (e) 0 (f) no conclusion

SOLUTION

(a) A number added to something that tends to ∞: the sum result gives ∞.

(b) $f(x) \to \infty$ and $g(x) \to -2 < 0$: the product result gives $-\infty$.

(c) Nonzero over zero: the top tends to −2 and the bottom tends to 0 through positive values. The signs are opposite, so the limit is $-\infty$.

(d) The shape $\infty \cdot 0$ decides nothing. With $f(x) = \dfrac{1}{(x-4)^2}$ and $h(x) = (x-4)^2$ the product is 1; with the same f and $h(x) = (x-4)^4$ it tends to 0. No conclusion.

(e) None of the results covers a number over something that grows without bound, so argue directly, as the lesson does for $\dfrac{1}{\ln x}$ as $x \to 0^+$. Since $g(x) \to -2$, close enough to 4 the value $g(x)$ is within 1 of −2, so $|g(x)| < 3$; and for any $M > 0$, $f(x) > M$ close enough to 4. Then $\left|\dfrac{g(x)}{f(x)}\right| < \dfrac{3}{M}$, which can be made as small as we like, so the limit is 0.

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(f) Both $f(x)$ and $\dfrac{1}{h(x)}$ tend to ∞ (the second by nonzero over zero), so this is the shape $\infty - \infty$, which decides nothing. With $f(x) = \dfrac{1}{(x-4)^2}$ and $h(x) = (x-4)^2$ the difference is 0; with $f(x) = \dfrac{2}{(x-4)^2}$ and the same h it is $\dfrac{1}{(x-4)^2} \to \infty$. No conclusion.

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