Derivatives of polynomials and powers

Taking a limit for every derivative is slow. Four short rules give the derivative of any polynomial, and of many roots and fractions, in a line or two.

WORDS TO KNOW

$\frac{d}{dx}\big[\ \cdot\ \big]$**:** "the derivative with respect to x of" whatever is in the brackets.

Coefficient: the constant that multiplies a power, such as the -6 in $-6x^4$.

$\left.\frac{dy}{dx}\right|_{x=a}$**:** the derivative worked out at $x = a$. It is the same number as $f'(a)$.

Constant: a number with no variable in it, such as 9, π^2 or $\sqrt{5}$.

Polynomial: a sum of constants times positive whole-number powers of x, plus a constant.

KEY IDEAS

1. Four rules, each proved once from the limit

Each rule comes from the definition $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. After that, you never take the limit again for these functions.

Constant: $\frac{d}{dx}\big[c\big] = 0$. A horizontal line has slope 0.

Constant multiple: $\frac{d}{dx}\big[kf(x)\big] = kf'(x)$. The coefficient stays in front.

Sum and difference: $\frac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$, where f' and g' exist.

Power: $\frac{d}{dx}\big[x^n\big] = nx^{n-1}$. The exponent comes down in front and drops by one.

2. A polynomial goes term by term

Split the polynomial into terms. Keep each coefficient, use the power rule on each power, and send the constant term to 0. A polynomial of degree $n \geq 1$ has a derivative of degree $n - 1$. The rules say nothing yet about products or quotients, so rewrite first: expand a product, and split a fraction whose bottom is a single power of x.

Products: multiply out, then go term by term.

Fractions over one power of x: divide each term by it. This holds only for $x \neq 0$.

3. Any power of x, after a rewrite

$$\frac{d}{dx}\big[x^r\big] = rx^{r-1} \qquad (r \neq 0)$$

First write roots and fractions as powers: $\sqrt[q]{x^p} = x^{p/q}$ and $\frac{1}{x^r} = x^{-r}$. A coefficient stays outside: $\frac{7}{x^4} = 7x^{-4}$.

Negative powers need $x \neq 0$. An even root such as $\sqrt{x}$ has a derivative only for $x > 0$. At $x = 0$, a power $x^{p/q}$ with $0 < \frac{p}{q} < 1$ has no derivative: the graph has a cusp or a vertical tangent there.

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WATCH OUT

- **The derivative of a product is not the product of the derivatives.** Expand the product first (Example 2 shows the two answers differ).
- **A constant has derivative 0, even when it looks like a power.** π^2 and 5^3 have no x in them, so the power rule does not apply.
- **A number in the bottom stays in the bottom.** $\frac{1}{6x^2} = \frac{1}{6}x^{-2}$, not $6x^{-2}$.
- **Keep the domain.** If you divided by x, or the function has a root or a negative power, say for which x the derivative holds.

WORKED EXAMPLES

Example 1. Find $f'(x)$ for $f(x) = 2x^5 - 4x^3 + 6x - 1$, and find $f'(-1)$.

1. Term by term: $2 \cdot 5x^4 = 10x^4$, then $-4 \cdot 3x^2 = -12x^2$, then $6 \cdot 1 = 6$, and the constant -1 gives 0.
$$f'(x) = 10x^4 - 12x^2 + 6$$
2. At $x = -1$, $(-1)^4 = 1$ and $(-1)^2 = 1$, so $f'(-1) = 10 - 12 + 6 = 4$.

Answer: $f'(x) = 10x^4 - 12x^2 + 6$ and $f'(-1) = 4$. The degree drops from 5 to 4.

Example 2. Find the derivative of (a) $p(x) = (x + 3)(x^2 - 2)$ and (b) $q(x) = \dfrac{8x^3 - 2x^2 + 3}{x^2}$.

1. (a) Expand first: $p(x) = x^3 + 3x^2 - 2x - 6$. Term by term, $p'(x) = 3x^2 + 6x - 2$.
2. The product of the two derivatives would be $(1)(2x) = 2x$. At $x = 1$ that is 2, but $p'(1) = 3 + 6 - 2 = 7$. So that shortcut is wrong.
3. (b) The bottom is one power of x. For $x \neq 0$, divide each term by x^2: $q(x) = 8x - 2 + 3x^{-2}$.
4. Term by term, $q'(x) = 8 - 6x^{-3}$.

Answer: (a) $p'(x) = 3x^2 + 6x - 2$. (b) $q'(x) = 8 - \frac{6}{x^3}$, for $x \neq 0$.

Example 3. Let $y = 8\sqrt{x} + \dfrac{4}{x^3}$. Find $\dfrac{dy}{dx}$ and its value at $x = 4$.

1. Rewrite as powers: $y = 8x^{1/2} + 4x^{-3}$.
2. Power rule on each term: $8 \cdot \frac{1}{2}x^{-1/2} = 4x^{-1/2}$, and $4 \cdot (-3)x^{-4} = -12x^{-4}$.
$$\frac{dy}{dx} = \frac{4}{\sqrt{x}} - \frac{12}{x^4}, \qquad x > 0$$
3. The square root needs $x \geq 0$, and there is no derivative at 0, so the answer holds for $x > 0$.
4. At $x = 4$: $\frac{4}{2} - \frac{12}{256} = 2 - \frac{3}{64} = \frac{125}{64}$.

Answer: $\frac{dy}{dx} = \frac{4}{\sqrt{x}} - \frac{12}{x^4}$ for $x > 0$, and its value at $x = 4$ is $\frac{125}{64}$.

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