Calculus 1 Unit 3 worksheet: The power rule and basic differentiation rules (answer key)

ANSWERS AT A GLANCE

1. (a) $15x^{14}$ (b) $-6x^8$ (c) $20t^4 - 2t$ (d) $2x^5 - 10x$ (e) $12x^3 - 21x^2 + x - 1$ (f) $3\sqrt{2}\,x^2$ (g) $3u^4 - 3u$ (h) $6x^2 + 16x - 3$

2. (a) $-\dfrac{21}{x^4}$, for $x \neq 0$ (b) $\dfrac{2}{5}x^{-3/5} = \dfrac{2}{5\sqrt[5]{x^3}}$, for $x \neq 0$ (c) $-\dfrac{1}{2x^3}$, for $x \neq 0$ (d) $\dfrac{5}{2}x^{3/2}$, for $x > 0$ (e) $3 + \dfrac{2}{x^2} - \dfrac{10}{x^3}$, for $x \neq 0$

3. $a = -3$ and $b = -4$

4. (a) $-\dfrac{10}{x^3}$, for $x \neq 0$ (b) $\dfrac{3}{2}\sqrt{x}$, for $x > 0$ (c) $-3x^{-3/2}$, for $x > 0$ (d) $-\dfrac{3}{2x^4}$, for $x \neq 0$

5. (a) $f'(x) = \dfrac{5}{3}x^{2/3} - \dfrac{10}{3}x^{-1/3} = \dfrac{5(x-2)}{3\sqrt[3]{x}}$ for $x \neq 0$ (b) $f'(8) = 5$ and $f'(-1) = 5$ (c) $x = 2$ (d) $f'(0)$ does not exist: the quotient $h^{2/3} - 5h^{-1/3}$ decreases without bound as $h \to 0^+$ and grows without bound as $h \to 0^-$.

WORKED SOLUTIONS

1. Find each derivative with the power, constant multiple, sum and constant rules. For a function written $y = \ldots$, give $\dfrac{dy}{dx}$.

(a) $y = x^{15}$

(b) $y = -\dfrac{2}{3}x^9$

(c) $\dfrac{d}{dt}\left[4t^5 - t^2\right]$

(d) $y = \dfrac{x^6}{3} - 5x^2 + 8$

(e) $y = 3x^4 - 7x^3 + \dfrac{1}{2}x^2 - x + 12$

(f) $y = \sqrt{2}\,x^3 + \pi^2$

(g) $\dfrac{d}{du}\left[0.6u^5 - 1.5u^2\right]$

(h) $y = (2x^2 - 3)(x + 4)$

ANSWER

(a) $15x^{14}$ (b) $-6x^8$ (c) $20t^4 - 2t$ (d) $2x^5 - 10x$ (e) $12x^3 - 21x^2 + x - 1$ (f) $3\sqrt{2}\,x^2$ (g) $3u^4 - 3u$ (h) $6x^2 + 16x - 3$

SOLUTION

(a) The exponent comes down and drops by one: $15x^{14}$.

(b) The constant multiple rule keeps $-\dfrac{2}{3}$: $-\dfrac{2}{3} \cdot 9x^8 = -6x^8$.

(c) The variable is t: $4 \cdot 5t^4 - 2t = 20t^4 - 2t$.

(d) $\dfrac{x^6}{3} = \dfrac{1}{3}x^6$, so its derivative is $\dfrac{1}{3} \cdot 6x^5 = 2x^5$; then $-5 \cdot 2x = -10x$, and the constant 8 gives 0: $2x^5 - 10x$.

(e) Term by term: $3 \cdot 4x^3 = 12x^3$, $-7 \cdot 3x^2 = -21x^2$, $\dfrac{1}{2} \cdot 2x = x$, -1, and 0 for the constant: $12x^3 - 21x^2 + x - 1$.

(f) $\sqrt{2}$ is a constant multiple, and π^2 is a constant with no x in it, so its derivative is 0: $3\sqrt{2}\,x^2$.

(g) The variable is u: $0.6 \cdot 5u^4 - 1.5 \cdot 2u = 3u^4 - 3u$.

(h) The rules cover sums, not products, so expand first: $(2x^2 - 3)(x + 4) = 2x^3 + 8x^2 - 3x - 12$. Term by term, $\dfrac{dy}{dx} = 6x^2 + 16x - 3$.

2. Rewrite each function as a sum of constant multiples of powers of x, then find $\dfrac{dy}{dx}$. Say for which x your answer holds.

(a) $y = \dfrac{7}{x^3}$

(b) $y = \sqrt[5]{x^2}$

(c) $y = \dfrac{1}{4x^2}$

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(d) $y = x^2\sqrt{x}$ (e) $y = \dfrac{3x^3 - 2x + 5}{x^2}$

ANSWER

(a) $-\dfrac{21}{x^4}$, for $x \neq 0$ (b) $\dfrac{2}{5}x^{-3/5} = \dfrac{2}{5\sqrt[5]{x^3}}$, for $x \neq 0$ (c) $-\dfrac{1}{2x^3}$, for $x \neq 0$ (d) $\dfrac{5}{2}x^{3/2}$, for $x > 0$ (e) $3 + \dfrac{2}{x^2} - \dfrac{10}{x^3}$, for $x \neq 0$

SOLUTION

(a) $y = 7x^{-3}$, so $\dfrac{dy}{dx} = 7 \cdot (-3)x^{-4} = -21x^{-4} = -\dfrac{21}{x^4}$, for $x \neq 0$.

(b) $y = x^{2/5}$, defined for every real x because fifth roots of negative numbers exist. The power rule gives $\dfrac{2}{5}x^{-3/5} = \dfrac{2}{5\sqrt[5]{x^3}}$, for $x \neq 0$. At 0 the exponent $\dfrac{2}{5}$ is between 0 and 1, so there is no derivative there (a cusp).

(c) The 4 is not raised to the power: $y = \dfrac{1}{4}x^{-2}$, so $\dfrac{dy}{dx} = \dfrac{1}{4} \cdot (-2)x^{-3} = -\dfrac{1}{2x^3}$, for $x \neq 0$.

(d) $x^2\sqrt{x} = x^2 \cdot x^{1/2} = x^{5/2}$, so $\dfrac{dy}{dx} = \dfrac{5}{2}x^{3/2}$. The function is defined only for $x \geq 0$, and at 0 there is no room to the left, so the answer holds for $x > 0$.

(e) The bottom is a single power of x, so split term by term: for $x \neq 0$, $y = 3x - 2x^{-1} + 5x^{-2}$. Then $\dfrac{dy}{dx} = 3 + 2x^{-2} - 10x^{-3} = 3 + \dfrac{2}{x^2} - \dfrac{10}{x^3}$, for $x \neq 0$.

3. The curve $y = ax^3 + bx^2$, where a and b are constants, has slope −17 at $x = 1$ and slope −52 at $x = 2$. Find a and b.

ANSWER

$a = -3$ and $b = -4$

SOLUTION

Differentiate term by term, treating a and b as constant multiples: $\dfrac{dy}{dx} = 3ax^2 + 2bx$. At $x = 1$ the slope is $3a + 2b$, and at $x = 2$ it is $12a + 4b$:

$$3a + 2b = -17, \qquad 12a + 4b = -52.$$

Twice the first equation is $6a + 4b = -34$. Subtract it from the second: $6a = -52 - (-34) = -18$, so $a = -3$. Then $2b = -17 - 3 \cdot (-3) = -8$, so $b = -4$.

4. Lucas rewrites each function as a power of x and then uses the power rule, but each line has one mistake. Find the mistake, and give the correct derivative and the x for which it holds.

(a) $\dfrac{d}{dx}\left[\dfrac{5}{x^2}\right] = \dfrac{5}{2x}$

(b) $\sqrt{x^3} = x^{2/3}$, so $\dfrac{d}{dx}\left[\sqrt{x^3}\right] = \dfrac{2}{3}x^{-1/3}$

(c) $\dfrac{6}{\sqrt{x}} = 6x^{-1/2}$, so $\dfrac{d}{dx}\left[\dfrac{6}{\sqrt{x}}\right] = -3x^{1/2}$

(d) $\dfrac{1}{2x^3} = 2x^{-3}$, so $\dfrac{d}{dx}\left[\dfrac{1}{2x^3}\right] = -6x^{-4}$

ANSWER

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(a) $-\dfrac{10}{x^3}$, for $x \neq 0$ (b) $\dfrac{3}{2}\sqrt{x}$, for $x > 0$ (c) $-3x^{-3/2}$, for $x > 0$ (d) $-\dfrac{3}{2x^4}$, for $x \neq 0$

SOLUTION

(a) He differentiated the bottom and left the 5 on top; there is no rule for a quotient like that. Rewrite first: $\dfrac{5}{x^2} = 5x^{-2}$, so the derivative is $5 \cdot (-2)x^{-3} = -\dfrac{10}{x^3}$, for $x \neq 0$.

(b) He swapped the parts of the exponent. The power inside the root goes on top and the index of the root on the bottom: $\sqrt{x^3} = x^{3/2}$. The derivative is $\dfrac{3}{2}x^{1/2} = \dfrac{3}{2}\sqrt{x}$, for $x > 0$ (the function needs $x \geq 0$, with no room to the left of 0).

(c) The rewrite is right, but the exponent must drop by one: $-\dfrac{1}{2} - 1 = -\dfrac{3}{2}$, not $+\dfrac{1}{2}$. The derivative is $6 \cdot \left(-\dfrac{1}{2}\right) x^{-3/2} = -3x^{-3/2}$, for $x > 0$.

(d) The 2 is in the bottom, so it stays a divisor: $\dfrac{1}{2x^3} = \dfrac{1}{2}x^{-3}$, not $2x^{-3}$. The derivative is $\dfrac{1}{2} \cdot (-3)x^{-4} = -\dfrac{3}{2x^4}$, for $x \neq 0$.

5. Let $f(x) = x^{5/3} - 5x^{2/3}$, which is defined for every real x because cube roots of negative numbers exist.

(a) Find $f'(x)$ for $x \neq 0$, and write it as a single fraction.

(b) Find $f'(8)$ and $f'(-1)$.

(c) Find every x at which $f'(x) = 0$.

(d) Use the difference quotient at 0 to decide whether $f'(0)$ exists.

ANSWER

(a) $f'(x) = \dfrac{5}{3}x^{2/3} - \dfrac{10}{3}x^{-1/3} = \dfrac{5(x-2)}{3\sqrt[3]{x}}$ for $x \neq 0$ (b) $f'(8) = 5$ and $f'(-1) = 5$ (c) $x = 2$ (d) $f'(0)$ does not exist: the quotient $h^{2/3} - 5h^{-1/3}$ decreases without bound as $h \to 0^+$ and grows without bound as $h \to 0^-$.

SOLUTION

(a) The power rule on each term (rational exponents, as the lesson states the rule) and the constant multiple rule:

$$f'(x) = \frac{5}{3}x^{2/3} - 5 \cdot \frac{2}{3}x^{-1/3} = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}.$$

Factor out $\dfrac{5}{3}x^{-1/3}$, using $x^{2/3} = x \cdot x^{-1/3}$:

$$f'(x) = \frac{5}{3}x^{-1/3}(x-2) = \frac{5(x-2)}{3\sqrt[3]{x}}, \qquad x \neq 0.$$

(b) $\sqrt[3]{8} = 2$, so $f'(8) = \dfrac{5 \cdot 6}{3 \cdot 2} = 5$. $\sqrt[3]{-1} = -1$, so $f'(-1) = \dfrac{5 \cdot (-3)}{3 \cdot (-1)} = 5$.

(c) A fraction is 0 when its top is 0 and its bottom is not: $x - 2 = 0$, so $x = 2$. (At $x = 0$ the formula is undefined, not zero.)

(d) Since $f(0) = 0$,

$$\frac{f(0+h) - f(0)}{h} = \frac{h^{5/3} - 5h^{2/3}}{h} = h^{2/3} - \frac{5}{\sqrt[3]{h}}.$$

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As $h \to 0$ the first term tends to 0. As $h \to 0^+$, $\sqrt[3]{h}$ is a small positive number, so $-\dfrac{5}{\sqrt[3]{h}}$ decreases without bound; as $h \to 0^-$ it is a small negative number, so the term grows without bound. The quotient has no limit, so $f'(0)$ does not exist (the graph has a cusp at the origin).

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