The chain rule

When one quantity depends on a second, and the second depends on a third, how fast does the first change when the third does?

WORDS TO KNOW

Composition $f \circ g$: the function whose value at x is $f(g(x))$. The inner function g acts first, then the outer function f acts on its output.

Inside (inner function): the part fed into the outer rule, such as $x^2 + 9$ in $\sqrt{x^2 + 9}$.

Outer derivative: the derivative of the outer function, evaluated at the inside.

Inner derivative: the derivative of the inside, the rate at which the inside changes.

$\frac{dy}{du}$**:** the derivative of y with respect to u.

KEY IDEAS

1. Rates in a chain multiply

If u moves $\frac{du}{dx}$ units per unit of x, and y moves $\frac{dy}{du}$ units per unit of u, then y moves their product per unit of x, like gears in a row. Units multiply too: (kilopascals per meter) times (meters per minute) is kilopascals per minute.

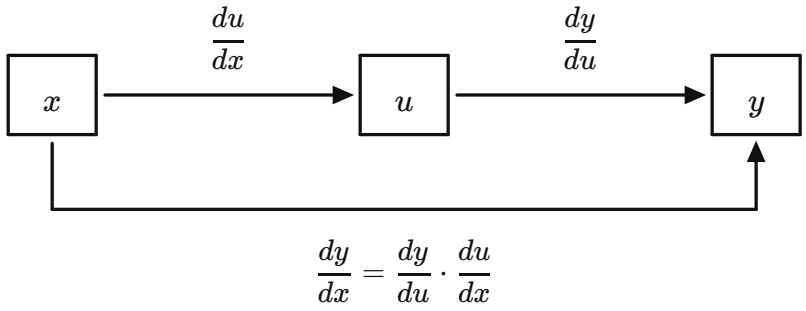
Chain rule. If g is differentiable at x and f is differentiable at $g(x)$, then

$$(f(g(x)))' = f'(g(x))\, g'(x).$$

In Leibniz notation, with $u = g(x)$ and $y = f(u)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \quad \text{with } \frac{dy}{du} \text{ read at } u = g(x).$$

The du seems to cancel. That is a good memory aid, but $\frac{dy}{du}$ is a limit, not a fraction.



2. The forms you will use most

Each form holds where g is differentiable at x and the outer function is differentiable at $g(x)$.

$$\frac{d}{dx}\left[g(x)^n\right] = n\, g(x)^{n-1}\, g'(x) \qquad \frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}\, g'(x)$$

$$\frac{d}{dx}\left[\sin g(x)\right] = \cos g(x)\; g'(x) \qquad \frac{d}{dx}\left[\cos g(x)\right] = -\sin g(x)\; g'(x)$$

$$\frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)} \ln a\; g'(x), \quad a > 0$$

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3. More layers, more factors

For $y = f(u)$, $u = g(v)$, $v = h(x)$, peel from the outside in, one factor per layer:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

WATCH OUT

- **Do not forget the inner derivative.** The derivative of $(x^2 + 9)^5$ is $5(x^2 + 9)^4 \cdot 2x$, not just $5(x^2 + 9)^4$.
- **The outer derivative is read at the inside.** It is $f'(g(x))$, not $f'(x)$. With a table, look up $g(a)$ first, then find f' at that value.
- **Both hypotheses matter.** If g has a corner at $f(a)$, the rule does not give the derivative of $g(f(x))$ at a. Check each side instead.
- **Domains carry over.** $\sqrt{x}$ has no derivative at 0, so $\sqrt{1 - x^2}$ is differentiable only for $-1 < x < 1$.

WORKED EXAMPLES

Example 1. Find the tangent line to $y = \sqrt{x^2 + 9}$ at $x = 4$.

1. Write $y = (x^2 + 9)^{1/2}$. The outer derivative is $\frac{1}{2}u^{-1/2}$ and the inner derivative is $2x$:
$$\frac{dy}{dx} = \tfrac{1}{2}(x^2 + 9)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + 9}}$$
2. At $x = 4$: $y = \sqrt{25} = 5$ and the slope is $\frac{4}{5}$.
3. The line through (4, 5): $y = 5 + \frac{4}{5}(x - 4)$.

Answer: $y = \frac{4}{5}x + \frac{9}{5}$. **Check:** at $x = 4$ the line gives $\frac{16}{5} + \frac{9}{5} = 5$.

Example 2. Use the table to find $(f \circ g)'(1)$ and the derivative of $g(x)^2$ at $x = 3$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	4	2	3	−1
3	0	5	2	6

1. $(f \circ g)'(1) = f'(g(1))\, g'(1)$. The inside value is $g(1) = 3$, so read $f'(3) = 5$, not $f'(1)$.
2. So $(f \circ g)'(1) = 5 \cdot (-1) = -5$.
3. By the power form, the derivative of $g(x)^2$ is $2g(x)\, g'(x)$. At $x = 3$: $2 \cdot 2 \cdot 6 = 24$.

Answer: $(f \circ g)'(1) = -5$, and the derivative of $g(x)^2$ at $x = 3$ is 24.

Example 3. Differentiate $y = \sin^3(2x)$ and find the slope at $x = \frac{\pi}{12}$.

1. $\sin^3(2x)$ means $(\sin 2x)^3$. The layers, from outside in: a cube, a sine, then $2x$.
2. One factor per layer:
$$y' = 3(\sin 2x)^2 \cdot \cos 2x \cdot 2 = 6\sin^2(2x)\cos(2x)$$
3. At $x = \frac{\pi}{12}$, $2x = \frac{\pi}{6}$, so $\sin 2x = \frac{1}{2}$ and $\cos 2x = \frac{\sqrt{3}}{2}$.

Answer: $y' = 6\sin^2(2x)\cos(2x)$, and the slope is $6 \cdot \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$.

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