Calculus 1 Unit 4 worksheet: The chain rule (answer key)

ANSWERS AT A GLANCE

1. (a) $\dfrac{dy}{dx} = -18(4-3x)^5$ (b) $f'(x) = -\dfrac{2(6x^2+1)}{(2x^3+x)^3}$ (c) $g'(x) = \dfrac{2x}{3(x^2+7)^{2/3}}$ (d) $\dfrac{dy}{dx} = (6x-1)e^{3x^2-x}$ (e) $h'(x) = -\ln 5 \, \sin x \cdot 5^{\cos x}$

2. $\dfrac{dy}{dx} = 42$

3. (a) $(f \circ g)'(x) = 2x \ln 2 \cdot 2^{x^2}$; $(g \circ f)'(x) = 2 \ln 2 \cdot 4^x$ (b) $(f \circ g)'(1) = 4 \ln 2$ and $(g \circ f)'(1) = 8 \ln 2$; yes, the two orders give different derivatives. (c) $(2^x)^2 = 4^x$, whose derivative is $4^x \ln 4 = 2 \ln 2 \cdot 4^x$, the same.

4. (a) He left out the inner derivative $2x$: the derivative is $8x(x^2+1)^3$. (b) He differentiated the exponent in place instead of multiplying by its derivative: the derivative is $\cos x \, e^{\sin x}$. (c) He put the inner derivative inside the outer derivative instead of multiplying by it: the derivative is $\dfrac{3x^2}{2\sqrt{x^3+1}}$.

5. (a) $\dfrac{dR}{dt} = 8 \ln 2 \approx 5.545$ micromoles per second, per minute (b) $16 \ln 2 \approx 11.090$ micromoles per second, per minute

6. (a) 8 (b) $\dfrac{1}{3}$ (c) $P'(1) = 65$

WORKED SOLUTIONS

1. Find each derivative with the chain rule.

(a) $y = (4-3x)^6$ (b) $f(x) = (2x^3+x)^{-2}$ (c) $g(x) = \sqrt[3]{x^2+7}$

(d) $y = e^{3x^2-x}$ (e) $h(x) = 5^{\cos x}$

ANSWER

(a) $\dfrac{dy}{dx} = -18(4-3x)^5$ (b) $f'(x) = -\dfrac{2(6x^2+1)}{(2x^3+x)^3}$ (c) $g'(x) = \dfrac{2x}{3(x^2+7)^{2/3}}$ (d) $\dfrac{dy}{dx} = (6x-1)e^{3x^2-x}$ (e) $h'(x) = -\ln 5 \, \sin x \cdot 5^{\cos x}$

SOLUTION

(a) Outside u^6, inside $u = 4-3x$ with $\dfrac{du}{dx} = -3$: $\dfrac{dy}{dx} = 6(4-3x)^5 \cdot (-3) = -18(4-3x)^5$.

(b) Outside u^{-2}, inside $u = 2x^3+x$ with $\dfrac{du}{dx} = 6x^2+1$:

$$f'(x) = -2(2x^3+x)^{-3}(6x^2+1) = -\frac{2(6x^2+1)}{(2x^3+x)^3}.$$

(c) Write $g(x) = (x^2+7)^{1/3}$. The outer derivative is $\dfrac{1}{3}u^{-2/3}$ and the inner derivative is $2x$:

$$g'(x) = \tfrac{1}{3}(x^2+7)^{-2/3} \cdot 2x = \frac{2x}{3(x^2+7)^{2/3}}.$$

(d) Outside e^u, inside $u = 3x^2-x$ with $\dfrac{du}{dx} = 6x-1$: $\dfrac{dy}{dx} = e^{3x^2-x}(6x-1)$.

(e) Outside 5^u, with derivative $5^u \ln 5$; inside $u = \cos x$ with derivative $-\sin x$: $h'(x) = 5^{\cos x} \ln 5 \cdot (-\sin x) = -\ln 5 \, \sin x \cdot 5^{\cos x}$.

2. Let $y = u^3 - 6u$, where $u = x^2 + 4x$. Use $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ to find $\dfrac{dy}{dx}$ when $x = -1$.

ANSWER

$\dfrac{dy}{dx} = 42$

SOLUTION

The two rates are $\dfrac{dy}{du} = 3u^2 - 6$ and $\dfrac{du}{dx} = 2x + 4$.

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When $x = -1$, the middle quantity is $u = -3$, so $\dfrac{dy}{du} = 3 \cdot (-3)^2 - 6 = 21$, and $\dfrac{du}{dx} = 2$.

So $\dfrac{dy}{dx} = 21 \cdot 2 = 42$. The rate $\dfrac{dy}{du}$ is read at the current value of u, not at $x = -1$.

3. Let $f(x) = 2^x$ and $g(x) = x^2$.

(a) Find $(f \circ g)'(x)$ and $(g \circ f)'(x)$.

(b) Find $(f \circ g)'(1)$ and $(g \circ f)'(1)$. Does the order of composition matter?

(c) Check $(g \circ f)'(x)$ a second way: write $(g \circ f)(x)$ as a single exponential with one base and differentiate it.

ANSWER

(a) $(f \circ g)'(x) = 2x \ln 2 \cdot 2^{x^2}$; $(g \circ f)'(x) = 2 \ln 2 \cdot 4^x$ (b) $(f \circ g)'(1) = 4 \ln 2$ and $(g \circ f)'(1) = 8 \ln 2$; yes, the two orders give different derivatives. (c) $(2^x)^2 = 4^x$, whose derivative is $4^x \ln 4 = 2 \ln 2 \cdot 4^x$, the same.

SOLUTION

(a) $(f \circ g)(x) = 2^{x^2}$: outside 2^u, with derivative $2^u \ln 2$, inside $u = x^2$ with derivative $2x$. So $(f \circ g)'(x) = 2^{x^2} \ln 2 \cdot 2x = 2x \ln 2 \cdot 2^{x^2}$.

$(g \circ f)(x) = (2^x)^2$: outside u^2, with derivative $2u$, inside $u = 2^x$ with derivative $2^x \ln 2$. So $(g \circ f)'(x) = 2 \cdot 2^x \cdot 2^x \ln 2 = 2 \ln 2 \cdot 4^x$.

(b) $(f \circ g)'(1) = 2 \ln 2 \cdot 2 = 4 \ln 2$ and $(g \circ f)'(1) = 2 \ln 2 \cdot 4 = 8 \ln 2$. They differ, so the order matters: $f \circ g$ and $g \circ f$ are different functions.

(c) $(2^x)^2 = 2^{2x} = 4^x$, and $\dfrac{d}{dx}\left[4^x\right] = 4^x \ln 4$. Since $\ln 4 = 2 \ln 2$, this is $2 \ln 2 \cdot 4^x$, which matches (a).

4. Ben wrote these three derivatives, and each one is wrong. For each, say what Ben did wrong and give the correct derivative.

(a) $\dfrac{d}{dx}\left[(x^2+1)^4\right] = 4(x^2+1)^3$

(b) $\dfrac{d}{dx}\left[e^{\sin x}\right] = e^{\cos x}$

(c) $\dfrac{d}{dx}\left[\sqrt{x^3+1}\right] = \dfrac{1}{2\sqrt{3x^2}}$

ANSWER

(a) He left out the inner derivative $2x$: the derivative is $8x(x^2+1)^3$. (b) He differentiated the exponent in place instead of multiplying by its derivative: the derivative is $\cos x \, e^{\sin x}$. (c) He put the inner derivative inside the outer derivative instead of multiplying by it: the derivative is $\dfrac{3x^2}{2\sqrt{x^3+1}}$.

SOLUTION

(a) The outer derivative $4u^3$ at $u = x^2 + 1$ is right, but the chain rule multiplies by the rate of the inside, $2x$: $4(x^2+1)^3 \cdot 2x = 8x(x^2+1)^3$.

(b) The outer function e^u is its own derivative, so the outer derivative at the inside is $e^{\sin x}$, unchanged. Then multiply by the inner derivative $\cos x$: $e^{\sin x} \cos x$. Ben changed the inside instead of multiplying by its rate.

(c) The outer derivative $\dfrac{1}{2\sqrt{u}}$ must be evaluated at the inside $u = x^3 + 1$, not at its derivative. Then multiply by $3x^2$:

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$$\frac{1}{2\sqrt{x^3+1}}\cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3+1}}, \quad x > -1.$$

5. In Maya's chemistry class, a made-up model says that a reaction runs at $R = 5\cdot 2^{(T-20)/10}$ micromoles per second when the solution is at T degrees Celsius. Maya warms the solution so that its temperature t minutes after she starts is $T = 20 + 4t$ degrees Celsius.

(a) Use $\dfrac{dR}{dt} = \dfrac{dR}{dT}\cdot\dfrac{dT}{dt}$ to find how fast the reaction rate is rising at $t = 5$. Give the exact value with units, then use a calculator and round to three decimal places.

(b) In a second run Maya warms the solution twice as fast, so $T = 20 + 8t$. How fast is the reaction rate rising at the moment the solution reaches 40 degrees Celsius?

ANSWER

(a) $\dfrac{dR}{dt} = 8\ln 2 \approx 5.545$ micromoles per second, per minute (b) $16\ln 2 \approx 11.090$ micromoles per second, per minute

SOLUTION

(a) The outer rate is

$$\frac{dR}{dT} = 5\cdot 2^{(T-20)/10}\ln 2\cdot\frac{1}{10} = \frac{\ln 2}{2}\cdot 2^{(T-20)/10}$$

micromoles per second per degree, and $\dfrac{dT}{dt} = 4$ degrees per minute. At $t = 5$ the temperature is $T = 40$, so $2^{(40-20)/10} = 4$ and $\dfrac{dR}{dT} = 2\ln 2$. Then

$$\frac{dR}{dt} = 2\ln 2\cdot 4 = 8\ln 2 \approx 5.545$$

micromoles per second, per minute.

(b) At 40 degrees $\dfrac{dR}{dT}$ is $2\ln 2$ again, since it depends only on T; now $\dfrac{dT}{dt} = 8$. So $\dfrac{dR}{dt} = 2\ln 2\cdot 8 = 16\ln 2 \approx 11.090$ micromoles per second, per minute: twice the rate in (a), because the middle quantity changes twice as fast.

6. The functions f and g are differentiable, with these values.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	4	−3	2	5
2	1	6	4	−2
4	9	2	1	3

(a) Find the derivative of $f(x^2)$ at $x = 2$.

(b) Find the derivative of $\sqrt{f(x)}$ at $x = 4$.

(c) Let $P(x) = g(x)\,f(g(x))$. Find $P'(1)$.

ANSWER

(a) 8 (b) $\dfrac{1}{3}$ (c) $P'(1) = 65$

SOLUTION

(a) The inside is x^2, with derivative $2x$. The derivative is $f'(x^2)\cdot 2x$; at $x = 2$ the inside is 4, so it is $f'(4)\cdot 4 = 2\cdot 4 = 8$.

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(b) The derivative of $f(x)^{1/2}$ is $\dfrac{1}{2} f(x)^{-1/2} f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$. At $x = 4$: $\dfrac{2}{2\sqrt{9}} = \dfrac{1}{3}$.

(c) The outermost operation is a product, so the product rule goes first and the chain rule handles $f(g(x))$:

$$P'(x) = g'(x)\, f(g(x)) + g(x)\, f'(g(x))\, g'(x).$$

At $x = 1$, $g(1) = 2$, so read $f(2) = 1$ and $f'(2) = 6$:

$$P'(1) = 5 \cdot 1 + 2 \cdot 6 \cdot 5 = 5 + 60 = 65.$$

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