u-substitution

The chain rule differentiates a function of a function. Run it backward, and you can integrate a function of an inside expression times the derivative of that inside.

WORDS TO KNOW

Inside function: the expression inside a power, a root, a trig function, an exponential or a denominator, such as x^2+9 in $\sqrt{x^2+9}$.

Differential $du = g'(x)\,dx$: what dx turns into when you change to u.

Change the limits: in a definite integral, replace each x-limit by its u-value.

Substitution $u = g(x)$: a new name for the inside function.

Back-substitute: put $g(x)$ back in place of u, so the answer is in x.

KEY IDEAS

1. The chain rule read backward

If $F' = f$, the chain rule gives $\frac{d}{dx}F(g(x)) = f(g(x))\,g'(x)$. Read the other way, that is an antiderivative.

$$\int f\big(g(x)\big)\,g'(x)\,dx = F\big(g(x)\big) + C$$

Steps: let $u = g(x)$ and find $du = g'(x)\,dx$. Write the whole integrand in u, with no x left. Integrate in u, add $+C$, back-substitute, and check by differentiating.

2. A constant can be fixed, a variable cannot

If $u = x^3 + 1$, then $du = 3x^2\,dx$. An integrand with only $x^2\,dx$ uses $x^2\,dx = \frac{1}{3}\,du$, because a constant moves through the integral sign. A leftover factor with x in it cannot move out. If a stray x is left, solve $u = g(x)$ for x (from $u = x + 3$, $x = u - 3$) and rewrite it in u.

Integrand	Choice	Antiderivative
$\big(g(x)\big)^n g'(x),\ n \neq -1$	$u = g(x)$	$\dfrac{\big(g(x)\big)^{n+1}}{n+1} + C$
$\dfrac{g'(x)}{g(x)}$	$u = g(x)$	$\ln\lvert g(x)\rvert + C$
$e^{g(x)}g'(x)$	$u = g(x)$	$e^{g(x)} + C$
$\sin^n x \cos x,\ n \neq -1$	$u = \sin x$	$\dfrac{\sin^{n+1} x}{n+1} + C$

3. Definite integrals: two routes that agree

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Either change the limits to u-values and finish in u, or find the antiderivative in x and use the x-limits. Never mix the two.

The new lower limit may be larger than the new upper one. That is allowed; swapping the limits changes the sign.

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WATCH OUT

- **Only a constant moves through the integral sign.** If the factor you need has an x in it, the choice of u fails, or the stray x must be written in u.
- **No x may remain.** An integral never holds x and u at once.
- **Limits belong to the variable.** A u-integral needs u-limits.
- **Keep the bars in** $\ln|u|$ unless $u > 0$ everywhere, as for $x^2 + 5$. The answer holds on each interval where $u \neq 0$.

WORKED EXAMPLES

Example 1. Find $\int x^3 \left(x^4 + 2\right)^5 \, dx$.

1. The inside is $x^4 + 2$, with derivative $4x^3$. Only x^3 is there, a constant factor off. Let $u = x^4 + 2$, so $du = 4x^3 \, dx$ and $x^3 \, dx = \frac{1}{4} \, du$.
2. Integrate in u:
$$\frac{1}{4} \int u^5 \, du = \frac{1}{4} \cdot \frac{u^6}{6} + C = \frac{u^6}{24} + C$$
3. Back-substitute.

Answer: $\frac{1}{24} \left(x^4 + 2\right)^6 + C$. **Check:** by the chain rule, $\frac{6}{24} \left(x^4 + 2\right)^5 \cdot 4x^3 = x^3 \left(x^4 + 2\right)^5$.

Example 2. Find $\int x(x-2)^4 \, dx$.

1. Let $u = x - 2$, so $du = dx$. A stray x is left, so solve for it: $x = u + 2$.
2. Now every x is in u:
$$\int (u+2)u^4 \, du = \int \left(u^5 + 2u^4\right) du = \frac{u^6}{6} + \frac{2u^5}{5} + C$$
3. Back-substitute.

Answer: $\frac{1}{6}(x-2)^6 + \frac{2}{5}(x-2)^5 + C$. **Check:** the derivative is $(x-2)^5 + 2(x-2)^4$, which factors as $(x-2)^4\big((x-2) + 2\big) = x(x-2)^4$.

Example 3. Evaluate $\int_0^{\pi/3} \sin x \cos^2 x \, dx$.

1. The inside is $\cos x$, with derivative $-\sin x$. Let $u = \cos x$, so $\sin x \, dx = -du$.
2. Change the limits: $x = 0$ gives $u = 1$, and $x = \frac{\pi}{3}$ gives $u = \frac{1}{2}$.
3. Swapping the limits absorbs the minus sign:
$$\int_1^{1/2} -u^2 \, du = \int_{1/2}^1 u^2 \, du = \frac{u^3}{3}\Big|_{1/2}^1 = \frac{1}{3} - \frac{1}{24} = \frac{7}{24}$$

Answer: $\frac{7}{24}$. **Check:** back-substitute instead. The antiderivative in x is $-\frac{1}{3}\cos^3 x$, which is $-\frac{1}{24}$ at $x = \frac{\pi}{3}$ and $-\frac{1}{3}$ at $x = 0$, and $-\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$. It is positive, as it must be, since the integrand is positive on $\left(0, \frac{\pi}{3}\right]$.

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