# Calculus 1 Unit 8 worksheet: u-substitution (answer key)

**ANSWERS AT A GLANCE**

1. (a) $-\dfrac{2}{(2x+5)^2}+C$, on $x>-\dfrac{5}{2}$ and on $x<-\dfrac{5}{2}$ separately (b) $\dfrac{1}{5}e^{x^5}+C$ (c) $\dfrac{\tan^4 x}{4}+C$, on each interval where $\cos x\neq 0$ (d) $\ln|\ln x|+C$, on $(0,1)$ and on $(1,\infty)$ separately
2. $-\dfrac{1}{4}\left(9-x^2\right)^6+C$
3. (a) $\dfrac{1}{10}(2x-1)^{5/2}+\dfrac{1}{2}(2x-1)^{3/2}+C$, for $x>\dfrac{1}{2}$ (b) $\dfrac{2}{7}(x+2)^{7/2}-\dfrac{8}{5}(x+2)^{5/2}+\dfrac{8}{3}(x+2)^{3/2}+C$, for $x>-2$
4. (a) $1$ (b) $\dfrac{14}{9}$
5. (a) $400\ln\dfrac{5}{3}$, about $204$ members (b) $M(T)=1{,}500+200\ln\dfrac{T^2+9}{9}$ (c) $T=3\sqrt{e^2-1}$, about $7.6$ weeks

**WORKED SOLUTIONS**

**1.** Find each indefinite integral by a substitution. Name $u$ and $du$, and check each answer by differentiating.

(a) $\displaystyle\int \frac{8}{(2x+5)^3}\,dx$ (b) $\displaystyle\int x^4 e^{x^5}\,dx$

(c) $\displaystyle\int \sec^2 x\tan^3 x\,dx$ (d) $\displaystyle\int \frac{dx}{x\ln x}$

**ANSWER**

(a) $-\dfrac{2}{(2x+5)^2}+C$, on $x>-\dfrac{5}{2}$ and on $x<-\dfrac{5}{2}$ separately (b) $\dfrac{1}{5}e^{x^5}+C$ (c) $\dfrac{\tan^4 x}{4}+C$, on each interval where $\cos x\neq 0$ (d) $\ln|\ln x|+C$, on $(0,1)$ and on $(1,\infty)$ separately

**SOLUTION**

(a) Let $u=2x+5$, so $du=2\,dx$ and $dx=\dfrac{1}{2}\,du$. Then $\displaystyle\int 8u^{-3}\cdot\frac{1}{2}\,du=4\cdot\frac{u^{-2}}{-2}+C=-\frac{2}{(2x+5)^2}+C$, on each side of $x=-\dfrac{5}{2}$. Check: $-2\cdot(-2)(2x+5)^{-3}\cdot 2=\dfrac{8}{(2x+5)^3}$.

(b) Let $u=x^5$, so $du=5x^4\,dx$ and $x^4\,dx=\dfrac{1}{5}\,du$. Then $\dfrac{1}{5}\displaystyle\int e^u\,du=\frac{1}{5}e^{x^5}+C$. Check: $\dfrac{1}{5}e^{x^5}\cdot 5x^4=x^4e^{x^5}$.

(c) Let $u=\tan x$, so $du=\sec^2 x\,dx$. Then $\displaystyle\int u^3\,du=\frac{u^4}{4}+C=\frac{\tan^4 x}{4}+C$, on each interval where $\cos x\neq 0$. Check: $\dfrac{4\tan^3 x}{4}\cdot\sec^2 x=\sec^2 x\tan^3 x$.

(d) Let $u=\ln x$, so $du=\dfrac{dx}{x}$. Then $\displaystyle\int\frac{du}{u}=\ln|u|+C=\ln|\ln x|+C$. The integrand needs $x>0$ and $\ln x\neq 0$, so the answer holds on $(0,1)$ and on $(1,\infty)$ separately. Check: $\dfrac{1}{\ln x}\cdot\dfrac{1}{x}=\dfrac{1}{x\ln x}$.

**2.** Find $\displaystyle\int 3x\left(9-x^2\right)^5\,dx$. Check your answer by differentiating.

**ANSWER**

$-\dfrac{1}{4}\left(9-x^2\right)^6+C$

**SOLUTION**

Let $u=9-x^2$, so $du=-2x\,dx$ and $3x\,dx=-\dfrac{3}{2}\,du$. The minus sign belongs to the constant:

$$\int 3x\left(9-x^2\right)^5\,dx=-\frac{3}{2}\int u^5\,du=-\frac{3}{2}\cdot\frac{u^6}{6}+C=-\frac{1}{4}u^6+C.$$


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Back-substitute: $-\frac{1}{4}\left(9-x^2\right)^6+C$.

Check: by the chain rule the derivative is $-\frac{1}{4}\cdot 6\left(9-x^2\right)^5\cdot(-2x)=3x\left(9-x^2\right)^5$.

**3.** In each integral the natural choice of $u$ leaves a stray factor of $x$. Solve the substitution for $x$, rewrite the whole integrand in $u$, and find the integral.

(a) $\displaystyle\int (x+1)\sqrt{2x-1}\,dx$

(b) $\displaystyle\int x^2\sqrt{x+2}\,dx$

**ANSWER**

(a) $\frac{1}{10}(2x-1)^{5/2}+\frac{1}{2}(2x-1)^{3/2}+C$, for $x>\frac{1}{2}$ (b) $\frac{2}{7}(x+2)^{7/2}-\frac{8}{5}(x+2)^{5/2}+\frac{8}{3}(x+2)^{3/2}+C$, for $x>-2$

**SOLUTION**

(a) Let $u=2x-1$, so $du=2\,dx$, $dx=\frac{1}{2}\,du$ and $x=\frac{u+1}{2}$. Then $x+1=\frac{u+3}{2}$, and

$$\int (x+1)\sqrt{2x-1}\,dx=\int \frac{u+3}{2}\,u^{1/2}\cdot\frac{1}{2}\,du=\frac{1}{4}\int\left(u^{3/2}+3u^{1/2}\right)du=\frac{1}{4}\left(\frac{2}{5}u^{5/2}+2u^{3/2}\right)+C.$$

So the integral is $\frac{1}{10}(2x-1)^{5/2}+\frac{1}{2}(2x-1)^{3/2}+C$ for $x>\frac{1}{2}$. Check: the derivative is $\frac{1}{2}(2x-1)^{3/2}+\frac{3}{2}(2x-1)^{1/2}=(2x-1)^{1/2}\left(\frac{2x-1}{2}+\frac{3}{2}\right)=(x+1)\sqrt{2x-1}$.

(b) Let $u=x+2$, so $du=dx$ and $x=u-2$. The stray factor is $x^2=(u-2)^2=u^2-4u+4$:

$$\int x^2\sqrt{x+2}\,dx=\int\left(u^{5/2}-4u^{3/2}+4u^{1/2}\right)du=\frac{2}{7}u^{7/2}-\frac{8}{5}u^{5/2}+\frac{8}{3}u^{3/2}+C.$$

So the integral is $\frac{2}{7}(x+2)^{7/2}-\frac{8}{5}(x+2)^{5/2}+\frac{8}{3}(x+2)^{3/2}+C$ for $x>-2$. Check: the derivative is $(x+2)^{5/2}-4(x+2)^{3/2}+4(x+2)^{1/2}=(x+2)^{1/2}\big((x+2)-2\big)^2=x^2\sqrt{x+2}$.

**4.** Evaluate each integral by a substitution, changing the limits of integration to $u$-values.

(a) $\displaystyle\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)dx$

(b) $\displaystyle\int_1^e \frac{\sqrt{1+3\ln x}}{x}\,dx$

**ANSWER**

(a) $1$ (b) $\frac{14}{9}$

**SOLUTION**

(a) Let $u=x^2$, so $du=2x\,dx$ and $x\,dx=\frac{1}{2}\,du$. The limits: $x=0$ gives $u=0$, and $x=\sqrt{\pi}$ gives $u=\pi$:

$$\int_0^{\sqrt{\pi}} x\sin\left(x^2\right)dx=\frac{1}{2}\int_0^{\pi}\sin u\,du=\frac{1}{2}\left(-\cos u\right)\Big|_0^{\pi}=\frac{1}{2}(1+1)=1.$$


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(b) Let $u = 1 + 3\ln x$, so $du = \dfrac{3}{x}\,dx$ and $\dfrac{dx}{x} = \dfrac{1}{3}\,du$. The limits: $x = 1$ gives $u = 1$, and $x = e$ gives $u = 4$:

$$\int_1^e \frac{\sqrt{1 + 3\ln x}}{x}\,dx = \frac{1}{3}\int_1^4 u^{1/2}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{9}(8 - 1) = \frac{14}{9}.$$

**5.** Ava runs an online chess club. When she opens a new season of tournaments, the club already has 1,500 members, and new members join at the rate

$$r(t) = \frac{400t}{t^2 + 9}$$

members per week, where $t$ is the number of weeks since the season opened. Nobody leaves the club during the season.

(a) Find the exact number of members who join in the first 4 weeks, and round it to the nearest whole member.

(b) Find a formula for $M(T)$, the number of members $T$ weeks after the season opens.

(c) After how many weeks have 400 new members joined? Give the exact value, then use a calculator and round to one decimal place.

**ANSWER**

(a) $400\ln\dfrac{5}{3}$, about 204 members (b) $M(T) = 1{,}500 + 200\ln\dfrac{T^2 + 9}{9}$ (c) $T = 3\sqrt{e^2 - 1}$, about 7.6 weeks

**SOLUTION**

(a) The number who join is the integral of the rate. Let $u = t^2 + 9$, so $du = 2t\,dt$ and $400t\,dt = 200\,du$. The limits: $t = 0$ gives $u = 9$, and $t = 4$ gives $u = 25$:

$$\int_0^4 \frac{400t}{t^2 + 9}\,dt = 200\int_9^{25}\frac{du}{u} = 200\left(\ln 25 - \ln 9\right) = 200\ln\frac{25}{9} = 400\ln\frac{5}{3}.$$

No bars are needed on $\ln u$ because $u > 0$ throughout. This is about 204.3, so about 204 members join.

(b) The same substitution with upper limit $u = T^2 + 9$ gives

$$M(T) = 1{,}500 + \int_0^T r(t)\,dt = 1{,}500 + 200\left(\ln\left(T^2 + 9\right) - \ln 9\right) = 1{,}500 + 200\ln\frac{T^2 + 9}{9}.$$

Check: $M(0) = 1{,}500$, and $M'(T) = 200\cdot\dfrac{2T}{T^2 + 9} = r(T)$.

(c) Set the new members equal to 400: $200\ln\dfrac{T^2 + 9}{9} = 400$, so $\dfrac{T^2 + 9}{9} = e^2$, $T^2 = 9e^2 - 9$ and, since $T > 0$, $T = 3\sqrt{e^2 - 1} \approx 7.6$ weeks.


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