Choosing an integration technique

An integral met in the wild carries no label. How do you read its shape, pick the shortest route, and know when no formula exists at all?

WORDS TO KNOW

Rewrite: algebra that replaces the integrand by an equal function, such as expanding, canceling, dividing or using an identity.

Route: the chain of methods you use, such as a substitution followed by parts.

Elementary function: a function built from constants and x by adding, subtracting, multiplying, dividing, taking roots, and composing with e^x, $\ln x$, trig and inverse trig functions.

Accumulation function: $F(x) = \int_a^x f(t)\,dt$. When f is continuous, its derivative is f.

KEY IDEAS

1. Rewrite before you choose

Try first: expand a product, split a fraction over its denominator, divide a top-heavy rational function, cancel a common factor, or use an identity such as $\sin^2 x + \cos^2 x = 1$ or $1 + \cos 2x = 2\cos^2 x$. The integrand stays the same function on the interval, so the integral does not change.

2. Read the shape

Several rows can fit one integral. Look at the first row first, because it is the cheapest.

The integrand shows	First move to try
a function and its derivative as a factor, up to a constant	u-substitution
two unlike kinds multiplied: a power times e^{kx} or a trig function, a log or inverse trig times a power	parts: differentiate the factor that simplifies
powers of sin and cos, or of sec and tan	trig identities
$\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ with no partner factor	trigonometric substitution
a rational function	divide if top-heavy, factor, then partial fractions or complete the square
a root of a linear expression, $e^{\sqrt{x}}$, or $\ln x$ inside something else	a substitution that removes it, then the next method

3. Chains, and one check for every route

A substitution may only change the kind of integral. With $t = \sqrt{x}$, $x = t^2$ and $dx = 2t\,dt$, a root becomes a power of t, and the new integral may need parts or partial fractions.

Two correct routes can cost very different amounts of work, but their answers differ only by a constant on each interval. Differentiate your answer: it must give back the integrand.

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4. Some antiderivatives have no formula

Every function continuous on an interval has an antiderivative there, the accumulation function. But it may not be elementary. e^{-x^2}, $\frac{\sin x}{x}$, $\frac{e^x}{x}$ and $\cos(x^2)$ have no elementary antiderivatives. Small changes matter: $\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + C$ is a one-line substitution.

What to do instead: find a number with the trapezoidal rule or Simpson's rule; study the accumulation function through its derivative $F' = f$; or integrate a power series term by term.

WATCH OUT

- **Do not skip the cheap look.** A rational function can always go to partial fractions, but if the top is the derivative of the bottom, the answer is one logarithm.
- **A root does not always mean trig substitution.** In $x^3\sqrt{9-x^2}$, the factor $x\,dx$ is part of du for $u = 9 - x^2$.
- **"Not elementary" does not mean "no antiderivative."** It means no formula built from the usual functions.

WORKED EXAMPLES

Example 1. Find $\displaystyle\int \frac{3x^2 - 4}{x^3 - 4x}\,dx$.

1. Since $x^3 - 4x = x(x-2)(x+2)$, partial fractions would work, with three constants to find.
2. But the top, $3x^2 - 4$, is the derivative of the bottom. Let $u = x^3 - 4x$, so $du = \left(3x^2 - 4\right)dx$ and the integral is $\int \frac{du}{u} = \ln|u| + C$.

Answer: $\ln\left|x^3 - 4x\right| + C$, on each interval that avoids -2, 0 and 2. **Check:** the chain rule gives $\frac{3x^2-4}{x^3-4x}$.

Example 2. Find $\displaystyle\int \frac{x^3 + 1}{x + 1}\,dx$.

1. Rewrite first. The top is a sum of cubes, $x^3 + 1 = (x+1)\left(x^2 - x + 1\right)$.
2. Wherever $x \neq -1$, cancel: the integrand is $x^2 - x + 1$. Integrate term by term.

Answer: $\frac{x^3}{3} - \frac{x^2}{2} + x + C$, on $x < -1$ and on $x > -1$. **Check:** the derivative is $x^2 - x + 1$.

Example 3. Find $\displaystyle\int \frac{\sqrt{x}}{x - 1}\,dx$.

1. The root is the obstacle, so remove it. Let $t = \sqrt{x}$ with $t \geq 0$, so $x = t^2$ and $dx = 2t\,dt$:
$$\int \frac{t}{t^2 - 1}\,2t\,dt = 2\int \frac{t^2}{t^2 - 1}\,dt$$
2. The fraction is top-heavy, so divide, then split the remainder by cover-up:
$$\frac{t^2}{t^2 - 1} = 1 + \frac{1}{t^2 - 1} = 1 + \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right)$$
3. Integrate, then put $t = \sqrt{x}$ back: $2t + \ln|t-1| - \ln|t+1| + C$.

Answer: $2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$, on $0 < x < 1$ and on $x > 1$. Three moves were chained: a substitution, division, and partial fractions.

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