Integration by parts

The product rule tells you how to differentiate a product. Read backward, it lets you integrate a product such as $x\cos 5x$, or a lone $\ln x$, that no substitution can reach.

WORDS TO KNOW

Parts: the two pieces you split the integrand into. u is the piece you differentiate, and dv is the piece you integrate (the dx always goes with dv).

$du = u'(x)\,dx$: the differential of u.

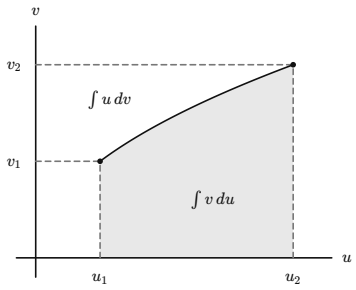
Tabular integration: a table that keeps track of repeated parts steps when u is a polynomial.

Reduction formula: a rule that writes an integral with a power n in terms of the same integral with power $n-1$, so each use lowers the power by one.

KEY IDEAS

1. The product rule, read backward

The product rule says $(uv)' = u'v + uv'$. Integrate both sides and move one integral across, and you get a trade: the integral you want for a different one.



$$\int u\,dv = uv - \int v\,du$$

On $[a,b]$: $\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du$.

This needs u and v to have continuous derivatives. Any antiderivative of dv can be v, so you never add a constant to v. Write $+C$ once, at the end.

2. Pick u so the new integral is simpler

The formula holds for every split, but it only helps when $\int v\,du$ is easier than the integral you began with.

u: a factor that differentiating makes simpler. A polynomial drops a degree. $\ln x$, $\arctan x$ and $\arcsin x$ turn into the algebraic $\frac{1}{x}$, $\frac{1}{1+x^2}$ and $\frac{1}{\sqrt{1-x^2}}$.

dv: a factor you can integrate, such as e^{kx}, $\sin kx$, $\cos kx$, a power of x, or plain dx. A log or an inverse trig function is nearly always u.

3. Repeat: a table, or a loop

Table: when u is a polynomial, list u and its derivatives down to 0, and dv and its antiderivatives beside them. Multiply each entry by the entry one row down in the other column, with signs $+, -, +, -, \ldots$

Loop: for $e^{ax}\sin bx$ neither factor ever dies out. Two steps bring back the integral I itself. Solve the equation for I.

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WATCH OUT

- **The wrong split makes things worse.** With $u = \cos 5x$ and $dv = x\,dx$, the new integral holds x^2. Choose u so the power goes down.
- **Keep the same kind of split in a loop.** If the second step makes the exponential u, it undoes the first step and you get $I = I$, which is true but empty.
- **A table needs a polynomial to differentiate.** The derivatives of $\ln x$ never reach 0. For a log, one step with $u = \ln x$ removes it.
- **Watch the domain.** An answer with $\ln x$ holds only for $x > 0$.

WORKED EXAMPLES

Example 1. Find $\int x^3 \ln x\,dx$.

1. The log is u: $u = \ln x$ and $dv = x^3\,dx$. So $du = \frac{1}{x}\,dx$ and $v = \frac{x^4}{4}$.
2. Use the formula. The x in du cancels one power of x, and the log is gone:

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

Answer: $\frac{x^4}{4}\ln x - \frac{x^4}{16} + C$, for $x > 0$. **Check:** the derivative is $x^3 \ln x + \frac{x^3}{4} - \frac{x^3}{4} = x^3 \ln x$.

Example 2. Find $\int x^2 \cos 3x\,dx$.

The factor x^2 dies out after three derivatives, so make a table. Differentiate x^2 down to 0, and integrate $\cos 3x$ again and again.

sign	x^2 and its derivatives	$\cos 3x$ and its antiderivatives
$+$	x^2	$\cos 3x$
$-$	$2x$	$\frac{1}{3}\sin 3x$
$+$	2	$-\frac{1}{9}\cos 3x$
	0	$-\frac{1}{27}\sin 3x$

1. Multiply each entry of the middle column by the entry one row down on the right, with its sign:

$$\int x^2 \cos 3x\,dx = \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x + C$$

Answer: $\frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x + C$. **Check:** in the derivative, every term but $x^2 \cos 3x$ cancels.

Example 3. Find $\int e^{3x} \cos x\,dx$.

1. Call it I. Take $u = \cos x$ and $dv = e^{3x}\,dx$, so $du = -\sin x\,dx$ and $v = \frac{1}{3}e^{3x}$:

$$I = \frac{1}{3}e^{3x}\cos x + \frac{1}{3}\int e^{3x}\sin x\,dx$$

2. Keep the exponential as dv: $u = \sin x$, $du = \cos x\,dx$:

$$\int e^{3x}\sin x\,dx = \frac{1}{3}e^{3x}\sin x - \frac{1}{3}I$$

3. Substitute: $I = \frac{1}{3}e^{3x}\cos x + \frac{1}{9}e^{3x}\sin x - \frac{1}{9}I$. Add $\frac{1}{9}I$ to both sides, so $\frac{10}{9}I = \frac{e^{3x}}{9}(3\cos x + \sin x)$.

Answer: $\frac{e^{3x}}{10}(3\cos x + \sin x) + C$. **Check:** the derivative is $\frac{e^{3x}}{10}(9\cos x + 3\sin x - 3\sin x + \cos x) = e^{3x}\cos x$.

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