Partial fractions

Adding fractions combines simple pieces into one fraction. To integrate a rational function, run that backward and split it into pieces you can integrate one at a time.

WORDS TO KNOW

Rational function: a polynomial over a polynomial, $\frac{P(x)}{Q(x)}$.

Partial fractions: the simple pieces a proper rational function splits into.

Irreducible quadratic: a quadratic with no real roots, so its discriminant $b^2 - 4ac$ is negative. It does not factor over the real numbers.

Proper: the degree of the top is less than the degree of the bottom.

Repeated factor: a factor that appears more than once, such as $(x-1)^2$.

KEY IDEAS

1. Proper first, then factor the bottom

If the top's degree is not less than the bottom's, divide first, and split only the proper remainder. Then factor the bottom into real linear factors and irreducible quadratics.

2. One piece for each factor

Factor of the bottom	Pieces it gives
$x - r$, once	$\dfrac{A}{x-r}$
$(x-r)^m$	$\dfrac{A_1}{x-r} + \dfrac{A_2}{(x-r)^2} + \cdots + \dfrac{A_m}{(x-r)^m}$
$x^2 + px + q$, irreducible	$\dfrac{Bx + C}{x^2 + px + q}$
$\left(x^2 + px + q\right)^m$	one $\dfrac{B_k x + C_k}{\left(x^2 + px + q\right)^k}$ for each $k = 1, \ldots, m$

The number of unknown constants always equals the degree of the bottom.

3. Find the constants

Clear the denominators. The result is an identity between two polynomials, true for every x.

Cover-up: to find the constant over $x - r$ (or over the highest power of a repeated factor), cover that factor and put $x = r$ into what is left.

Matching coefficients: set the coefficients of each power of x equal on both sides. This always works, and it finds the constants that cover-up cannot.

4. Integrate the pieces

$$\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C \qquad \int \frac{A}{(x-r)^k}\,dx = -\frac{A}{(k-1)(x-r)^{k-1}} + C \quad (k \geq 2)$$

For a quadratic piece, split the top into a multiple of $2x + p$ (a logarithm) plus a constant (an arctangent, after completing the square). For $a > 0$, $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$.

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WATCH OUT

- **Divide first if the fraction is not proper.** Cover-up still gives numbers, but their sum is a different function.
- **A repeated factor needs every power.** For $(x-1)^2$ you need both $\frac{A}{x-1}$ and $\frac{B}{(x-1)^2}$; the top power alone is not enough.
- **Irreducible means no real roots.** x^2-5 splits into $\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)$, so it is not an irreducible piece.
- **Answers hold between roots of the bottom.** A definite integral across a root of Q is improper.

WORKED EXAMPLES

Example 1. Find $\int \dfrac{x+5}{x^2-2x-3}\,dx$.

1. The fraction is proper, and $x^2-2x-3=(x-3)(x+1)$. So the form is $\frac{A}{x-3}+\frac{B}{x+1}$.
2. Cover-up for A: cover $x-3$ and put $x=3$ into $\frac{x+5}{x+1}$, which gives $A=\frac{8}{4}=2$.
3. Cover-up for B: cover $x+1$ and put $x=-1$ into $\frac{x+5}{x-3}$, which gives $B=\frac{4}{-4}=-1$.
4. Integrate each piece.

Answer: $2\ln|x-3|-\ln|x+1|+C$, on each interval that avoids -1 and 3. **Check:** $2(x+1)-(x-3)=x+5$.

Example 2. Find $\int \dfrac{3x^2+6}{(x+2)(x-1)^2}\,dx$.

1. The factor $x-1$ is repeated, so the form is $\frac{A}{x+2}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$.
2. Clear denominators: $3x^2+6=A(x-1)^2+B(x+2)(x-1)+C(x+2)$.
3. At $x=-2$: $18=9A$, so $A=2$. At $x=1$: $9=3C$, so $C=3$. No value isolates B, so match x^2: $3=A+B$, so $B=1$.
4. The piece $\frac{3}{(x-1)^2}$ integrates by the power rule to $-\frac{3}{x-1}$.

Answer: $2\ln|x+2|+\ln|x-1|-\frac{3}{x-1}+C$, on each interval that avoids -2 and 1. **Check:** $2(x-1)^2+(x+2)(x-1)+3(x+2)=3x^2+6$.

Example 3. Find $\int \dfrac{2x^2+6x+9}{x^3+9x}\,dx$.

1. $x^3+9x=x\left(x^2+9\right)$, and x^2+9 has no real roots. The form is $\frac{A}{x}+\frac{Bx+C}{x^2+9}$.
2. Clear denominators: $2x^2+6x+9=A\left(x^2+9\right)+(Bx+C)x$.
3. At $x=0$: $9=9A$, so $A=1$. Match x^2: $2=A+B$, so $B=1$. Match x: $C=6$.
4. Split the quadratic piece into a log part and an arctangent part:
$$\frac{x+6}{x^2+9}=\frac{1}{2}\cdot\frac{2x}{x^2+9}+\frac{6}{x^2+9}$$

Answer: $\ln|x|+\frac{1}{2}\ln\left(x^2+9\right)+2\arctan\frac{x}{3}+C$, on $x<0$ and on $x>0$. No bars are needed on x^2+9, which is always positive.

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