Trigonometric integrals

How do you integrate a product of trig powers such as $\sin^3 x \cos^4 x$ or $\tan^4 x \sec^4 x$? Use an identity to rewrite it until one factor is a du and the rest is a function of u.

WORDS TO KNOW

Set aside: keep one factor apart to become du, such as $\sin x\,dx$ for $u = \cos x$.

Power reducing formulas: $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$. They halve a power by doubling the angle.

Pythagorean identities: $\sin^2 x + \cos^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$. They trade an even power of one function for a polynomial in the other.

Product to sum formulas: rules that turn a product such as $\sin 3x \cos 5x$ into a sum of single sines or cosines.

KEY IDEAS

1. Sine and cosine: the odd power decides

Sine power odd: set one $\sin x$ aside, change the rest with $\sin^2 x = 1 - \cos^2 x$, and let $u = \cos x$, so $\sin x\,dx = -du$.

Cosine power odd: set one $\cos x$ aside, change the rest with $\cos^2 x = 1 - \sin^2 x$, and let $u = \sin x$.

Both powers even: no factor can be set aside. Lower the powers with the power reducing formulas, and repeat until each term is easy. The shortcut $\sin x \cos x = \frac{1}{2}\sin 2x$ often saves a round.

Exactly one factor goes to du, and what is left must be an even power, so the identity can change it. If both powers are odd, either route works, and the answers differ by a constant.

2. Different angles: product to sum

$$\sin A \cos B = \tfrac{1}{2}\big[\sin(A+B) + \sin(A-B)\big]$$

$$\sin A \sin B = \tfrac{1}{2}\big[\cos(A-B) - \cos(A+B)\big] \qquad \cos A \cos B = \tfrac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$$

3. Tangent and secant

Here $\frac{d}{dx}\tan x = \sec^2 x$ and $\frac{d}{dx}\sec x = \sec x \tan x$. Two more antiderivatives finish the leftovers: $\int \tan x\,dx = \ln|\sec x| + C$ and $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

Case	Set aside	u	Change the rest with
sine power odd	$\sin x\,dx$	$\cos x$	$\sin^2 x = 1 - \cos^2 x$
cosine power odd	$\cos x\,dx$	$\sin x$	$\cos^2 x = 1 - \sin^2 x$
secant power even	$\sec^2 x\,dx$	$\tan x$	$\sec^2 x = 1 + \tan^2 x$
tangent power odd (with a secant)	$\sec x \tan x\,dx$	$\sec x$	$\tan^2 x = \sec^2 x - 1$

Tangent alone: for $m \geq 2$, write $\tan^m x = \tan^{m-2} x \sec^2 x - \tan^{m-2} x$ and repeat on the second piece.

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WATCH OUT

- **Set aside only one factor, and only from an odd power.** Since $\sin x = \pm\sqrt{1-\cos^2 x}$, an odd power of sine is not a polynomial in $\cos x$. With both powers even, reduce the powers instead.
- **The angles must match.** These moves work for $\sin^m x \cos^n x$, one angle. In $\sin^2 x \cos 3x$ the angles differ, so rewrite first.
- **An odd power of secant with an even power of tangent,** such as $\int \sec^3 x\,dx$, fits no row. It needs integration by parts.
- **Tangent and secant answers hold** only on intervals where $\cos x \neq 0$.

WORKED EXAMPLES

Example 1. Find $\int \sin^3 x \cos^2 x\,dx$.

1. The sine power is odd. Set one $\sin x$ aside and change $\sin^2 x$ to $1-\cos^2 x$:
$$\int \left(1-\cos^2 x\right)\cos^2 x\,\sin x\,dx$$
2. Let $u = \cos x$, so $\sin x\,dx = -du$:
$$-\int \left(u^2-u^4\right)du = -\frac{u^3}{3}+\frac{u^5}{5}+C$$

Answer: $\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$. **Check:** the derivative is $-\cos^4 x \sin x + \cos^2 x \sin x = \sin x \cos^2 x\left(1-\cos^2 x\right)$, which is $\sin^3 x \cos^2 x$.

Example 2. Evaluate $\int_0^{\pi} \sin^4 x\,dx$.

1. The power is even, so lower it. Square the power reducing formula:
$$\sin^4 x = \frac{1}{4}\left(1-2\cos 2x+\cos^2 2x\right)$$
2. The last term is an even power again: $\cos^2 2x = \frac{1+\cos 4x}{2}$. So
$$\sin^4 x = \frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$
3. Integrate term by term. Both sine terms are 0 at 0 and at π:
$$\int_0^{\pi}\sin^4 x\,dx = \left[\frac{3x}{8}-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}\right]_0^{\pi} = \frac{3\pi}{8}$$

Answer: $\frac{3\pi}{8}$, about 1.18. **Check:** $0 \le \sin^4 x \le 1$, so the value must lie between 0 and π.

Example 3. Find $\int \tan^2 x \sec^6 x\,dx$.

1. The secant power is even. Set $\sec^2 x$ aside and change the other $\sec^4 x$ to $\left(1+\tan^2 x\right)^2$.
2. Let $u = \tan x$, so $du = \sec^2 x\,dx$:
$$\int u^2\left(1+u^2\right)^2 du = \int \left(u^2+2u^4+u^6\right)du = \frac{u^3}{3}+\frac{2u^5}{5}+\frac{u^7}{7}+C$$

Answer: $\frac{\tan^3 x}{3} + \frac{2\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$, on each interval where $\cos x \neq 0$. **Check:** the derivative is $\left(\tan^2 x + 2\tan^4 x + \tan^6 x\right)\sec^2 x = \tan^2 x\left(1+\tan^2 x\right)^2 \sec^2 x$, which is $\tan^2 x \sec^6 x$.

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