Trigonometric substitution

A root such as $\sqrt{25 - x^2}$ or $\sqrt{x^2 + 9}$ has no partner factor for a plain substitution. Set x equal to a trig function, and a Pythagorean identity turns the root into a single trig term.

WORDS TO KNOW

Inverse substitution: you name x as a function of a new variable, $x = g(\theta)$, with $dx = g'(\theta)\, d\theta$. It is allowed when g is one-to-one on the interval you use.

Completing the square: rewriting $x^2 + bx + c$ as $(x - h)^2 + k$.

Reference triangle: a right triangle built from the substitution, used to rewrite the answer in x.

Shifted square: a square such as $(x - 2)^2$. You substitute for $x - 2$ instead of x.

KEY IDEAS

1. Match the root to an identity

Each shape has its own substitution. The interval for θ makes the substitution one-to-one and lets you drop the absolute value bars.

Root	Substitution	Interval for θ	Root becomes
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$a \cos\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a \sec\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$	$a\lvert\tan\theta\rvert$

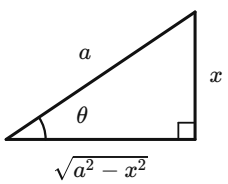
The identities: $1 - \sin^2\theta = \cos^2\theta$, $1 + \tan^2\theta = \sec^2\theta$, $\sec^2\theta - 1 = \tan^2\theta$.

Steps: write x and dx in θ, simplify the root, and integrate in θ with the methods of trigonometric integrals. Then go back to x.

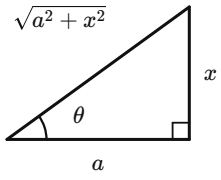
2. Back to x: the triangle, or new limits

Indefinite: draw the triangle. For $x = a \sin\theta$, the hypotenuse is a and the opposite side is x, so the adjacent side is $\sqrt{a^2 - x^2}$. Read off every ratio, and use $\theta = \arcsin\frac{x}{a}$.

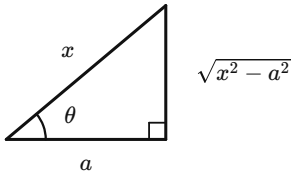
Definite: change each x-limit to its θ-value and finish in θ. You never return to x.



$x = a \sin\theta$



$x = a \tan\theta$



$x = a \sec\theta$

3. Secant has two pieces, and quadratics need a square first

$\sqrt{x^2 - a^2}$ lives on $x \ge a$ and on $x \le -a$. On the first piece the root is $a \tan\theta$; on the second it is $-a \tan\theta$, because $\tan\theta \le 0$ there. Work each piece separately. For a quadratic such as $x^2 - 4x + 13$, complete the square to get $(x - 2)^2 + 9$ and then let $x - 2 = 3 \tan\theta$. Here dx does not change.

Free sample from math.wiki for classroom use. Do not post. math.wiki

WATCH OUT

- **First look for something simpler.** In $\int \frac{x}{\sqrt{x^2+25}}\,dx$ the factor x is half the derivative of x^2+25, and a plain substitution takes one line.
- **The root of a square is an absolute value.** $\sqrt{a^2\cos^2\theta} = a|\cos\theta|$. Use the table's intervals, and read the sign of the root on the interval you use.
- **Answers hold only where the integrand is defined.** Nothing with $\sqrt{a^2-x^2}$ holds outside $[-a, a]$.
- **An integrand that blows up at a limit,** as in $\int_0^5 \frac{dx}{\sqrt{25-x^2}}$, makes an improper integral. It needs a limit.

WORKED EXAMPLES

Example 1. Evaluate $\int_0^1 \dfrac{x^2}{\sqrt{4-x^2}}\,dx$.

1. The root is $\sqrt{a^2-x^2}$ with $a=2$. Let $x = 2\sin\theta$, so $dx = 2\cos\theta\,d\theta$ and the root is $2\cos\theta$.
2. New limits: $x=0$ gives $\theta=0$, and $x=1$ gives $\sin\theta=\frac{1}{2}$, so $\theta=\frac{\pi}{6}$.
3. The root cancels, and the power reducing formula finishes:

$$\int_0^{\pi/6} \frac{4\sin^2\theta}{2\cos\theta}\,2\cos\theta\,d\theta = 4\int_0^{\pi/6}\sin^2\theta\,d\theta = 2\Big[\theta-\sin\theta\cos\theta\Big]_0^{\pi/6} = \frac{\pi}{3}-\frac{\sqrt{3}}{2}$$

Answer: $\frac{\pi}{3}-\frac{\sqrt{3}}{2}$, about 0.181. **Check:** the integrand rises from 0 to $\frac{1}{\sqrt{3}}\approx 0.58$ on an interval of length 1, so a value well under 0.58 makes sense.

Example 2. Find $\int \dfrac{dx}{x^2\sqrt{x^2+4}}$.

1. Let $x = 2\tan\theta$, with $\theta\neq 0$ since $x\neq 0$. Then $dx = 2\sec^2\theta\,d\theta$ and the root is $2\sec\theta$:

$$\int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\cdot 2\sec\theta} = \frac{1}{4}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

2. With $u=\sin\theta$ this is $\frac{1}{4}\int u^{-2}\,du = -\frac{1}{4\sin\theta}+C$.
3. The triangle has opposite side x, adjacent side 2 and hypotenuse $\sqrt{x^2+4}$, so $\sin\theta = \frac{x}{\sqrt{x^2+4}}$.

Answer: $-\frac{\sqrt{x^2+4}}{4x}+C$, on $x>0$ and on $x<0$ separately. **Check:** by the quotient rule,

$$\frac{d}{dx}\left(-\frac{\sqrt{x^2+4}}{4x}\right) = -\frac{1}{4}\cdot\frac{x^2-\left(x^2+4\right)}{x^2\sqrt{x^2+4}} = \frac{1}{x^2\sqrt{x^2+4}}$$

Example 3. Evaluate $\int_3^6 \dfrac{\sqrt{x^2-9}}{x}\,dx$.

1. Here $x\geq 3$, so let $x = 3\sec\theta$ with $0\leq\theta<\frac{\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\,d\theta$ and the root is $3\tan\theta$, since $\tan\theta\geq 0$.
2. New limits: $x=3$ gives $\sec\theta=1$, so $\theta=0$. $x=6$ gives $\sec\theta=2$, so $\theta=\frac{\pi}{3}$.
3. Integrate, using $\tan^2\theta = \sec^2\theta-1$:

$$\int_0^{\pi/3}\frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta = 3\int_0^{\pi/3}\left(\sec^2\theta-1\right)d\theta = 3\Big[\tan\theta-\theta\Big]_0^{\pi/3}$$

Answer: $3\sqrt{3}-\pi$, about 2.05. **Check:** the integrand rises from 0 to $\frac{\sqrt{27}}{6}\approx 0.87$ over an interval of length 3, so the value lies between 0 and about 2.6.

Free sample from math.wiki for classroom use. Do not post. math.wiki