Calculus 2 Unit 1 quiz 1: Integration by parts, Trigonometric integrals, Trigonometric substitution (answer key)

8 questions, 20 points

ANSWERS AT A GLANCE

1. $\dfrac{(x-3)\,5^x}{\ln 5} - \dfrac{5^x}{(\ln 5)^2} + C$
2. $\dfrac{\sin^4 2x}{8} - \dfrac{\sin^6 2x}{12} + C$
3. $\dfrac{\sec^{11} 4x}{44} - \dfrac{\sec^9 4x}{36} + C$
4. $\dfrac{1}{2}\arctan\dfrac{x}{3} + \dfrac{3x}{2\,(x^2+9)} + C$
5. $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$
6. $\dfrac{e^{3x}}{27}\left(9x^2 - 24x + 8\right) + C$
7. $\dfrac{\pi}{48} - \dfrac{\sqrt{3}}{64}$
8. $\dfrac{x}{\sqrt{16-x^2}} - \arcsin\dfrac{x}{4} + C$

WORKED SOLUTIONS

1. Find $\displaystyle\int (x-3)\,5^x\,dx$. (2 points)

ANSWER

$$\frac{(x-3)\,5^x}{\ln 5} - \frac{5^x}{(\ln 5)^2} + C$$

SOLUTION

Take $u = x - 3$, which differentiation turns into 1, and $dv = 5^x\,dx$. Then $du = dx$ and $v = \dfrac{5^x}{\ln 5}$:

$$\int (x-3)\,5^x\,dx = \frac{(x-3)\,5^x}{\ln 5} - \frac{1}{\ln 5}\int 5^x\,dx = \frac{(x-3)\,5^x}{\ln 5} - \frac{5^x}{(\ln 5)^2} + C.$$

Check: the derivative is $\dfrac{5^x}{\ln 5} + (x-3)\,5^x - \dfrac{5^x}{\ln 5} = (x-3)\,5^x$.

SCORING

1 point for $u = x - 3$ and $v = \dfrac{5^x}{\ln 5}$ in the parts formula; 1 point for the antiderivative with $+C$.

2. Find $\displaystyle\int \sin^3 2x \cos^3 2x\,dx$. (2 points)

ANSWER

$$\frac{\sin^4 2x}{8} - \frac{\sin^6 2x}{12} + C$$

SOLUTION

The power of cosine is odd, so set one $\cos 2x$ aside and write $\cos^2 2x = 1 - \sin^2 2x$. With $u = \sin 2x$, $du = 2\cos 2x\,dx$:

$$\int \sin^3 2x \cos^3 2x\,dx = \frac{1}{2}\int u^3\left(1-u^2\right)du = \frac{1}{2}\left(\frac{u^4}{4} - \frac{u^6}{6}\right) + C$$

$$= \frac{\sin^4 2x}{8} - \frac{\sin^6 2x}{12} + C.$$

SCORING

1 point for setting one $\cos 2x$ aside with $u = \sin 2x$ and $du = 2\cos 2x\,dx$; 1 point for the correct antiderivative, including the factor $\dfrac{1}{2}$ and $+C$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

3. Find $\int \tan^3 4x \sec^9 4x \, dx$. (2 points)

ANSWER

$$\frac{\sec^{11} 4x}{44} - \frac{\sec^9 4x}{36} + C$$

SOLUTION

The power of secant is odd, so the even-secant route is closed; the power of tangent is odd. Set $\sec 4x \tan 4x$ aside and write $\tan^2 4x = \sec^2 4x - 1$. With $u = \sec 4x$, $du = 4 \sec 4x \tan 4x \, dx$:

$$\int \tan^3 4x \sec^9 4x \, dx = \frac{1}{4} \int \left(u^2 - 1\right) u^8 \, du = \frac{1}{4} \left(\frac{u^{11}}{11} - \frac{u^9}{9} \right) + C$$

$$= \frac{\sec^{11} 4x}{44} - \frac{\sec^9 4x}{36} + C,$$

on each interval where $\cos 4x \neq 0$.

SCORING

1 point for setting $\sec 4x \tan 4x$ aside with $u = \sec 4x$ and converting $\tan^2$; 1 point for the correct antiderivative, including the factor $\frac{1}{4}$ and $+C$.

4. An integral was worked with the substitution $x = 3 \tan \theta$, and the result in θ is $\frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$. Write the result in terms of x. (2 points)

ANSWER

$$\frac{1}{2} \arctan \frac{x}{3} + \frac{3x}{2\left(x^2 + 9\right)} + C$$

SOLUTION

From $x = 3 \tan \theta$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\theta = \arctan \frac{x}{3}$. The triangle has opposite side x, adjacent side 3 and hypotenuse $\sqrt{x^2 + 9}$, so $\sin \theta = \frac{x}{\sqrt{x^2 + 9}}$ and $\cos \theta = \frac{3}{\sqrt{x^2 + 9}}$ (true for negative θ too). The triangle gives $\sin \theta$ and $\cos \theta$, not $\sin 2\theta$, so expand:

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \cdot \frac{x}{\sqrt{x^2 + 9}} \cdot \frac{3}{\sqrt{x^2 + 9}} = \frac{6x}{x^2 + 9}.$$

Then $\frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{1}{2} \arctan \frac{x}{3} + \frac{3x}{2\left(x^2 + 9\right)} + C$.

SCORING

1 point for $\sin 2\theta = 2 \sin \theta \cos \theta$ with both ratios read correctly from the triangle; 1 point for the final expression in x, including $\theta = \arctan \frac{x}{3}$.

5. Evaluate $\int_0^{\pi/4} x \sec^2 x \, dx$ exactly. You may use $\int \tan x \, dx = -\ln|\cos x| + C$. (3 points)

ANSWER

$$\frac{\pi}{4} - \frac{1}{2} \ln 2$$

SOLUTION

Free sample from math.wiki for classroom use. Do not post. math.wiki

Take $u = x$ and $dv = \sec^2 x\,dx$, so $du = dx$ and $v = \tan x$:

$$\int_0^{\pi/4} x\sec^2 x\,dx = \Big[x\tan x\Big]_0^{\pi/4} - \int_0^{\pi/4} \tan x\,dx = \frac{\pi}{4} - \Big[-\ln(\cos x)\Big]_0^{\pi/4}.$$

Since $\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$ and $\ln\dfrac{\sqrt{2}}{2} = -\dfrac{1}{2}\ln 2$, the last bracket is $\dfrac{1}{2}\ln 2 - 0$. So the integral is $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2 \approx 0.439$.

SCORING

1 point for $u = x$, $v = \tan x$ and the bracket $\dfrac{\pi}{4}$; 1 point for $\int \tan x\,dx = -\ln|\cos x|$ evaluated on $\left[0, \dfrac{\pi}{4}\right]$; 1 point for $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$.

6. Use tabular integration to find $\displaystyle\int (x^2 - 2x)e^{3x}\,dx$. Write the answer as e^{3x} times a polynomial. (3 points)

ANSWER

$$\frac{e^{3x}}{27}\left(9x^2 - 24x + 8\right) + C$$

SOLUTION

sign	$x^2 - 2x$ and its derivatives	e^{3x} and its antiderivatives
$+$	$x^2 - 2x$	e^{3x}
$-$	$2x - 2$	$\dfrac{1}{3}e^{3x}$
$+$	2	$\dfrac{1}{9}e^{3x}$
	0	$\dfrac{1}{27}e^{3x}$

$$\int (x^2 - 2x)e^{3x}\,dx = \frac{x^2 - 2x}{3}e^{3x} - \frac{2x - 2}{9}e^{3x} + \frac{2}{27}e^{3x} + C = \frac{e^{3x}}{27}\left(9x^2 - 24x + 8\right) + C.$$

Check: the derivative is $\dfrac{e^{3x}}{27}\left(27x^2 - 72x + 24 + 18x - 24\right) = e^{3x}\left(x^2 - 2x\right)$.

SCORING

1 point for the derivative column to 0 and the antiderivative column; 1 point for the alternating signs on the diagonal products; 1 point for $\dfrac{e^{3x}}{27}\left(9x^2 - 24x + 8\right) + C$.

7. Evaluate $\int_0^{\pi/6} \sin^2 x\cos^2 x\,dx$ exactly. (3 points)

ANSWER

$$\frac{\pi}{48} - \frac{\sqrt{3}}{64}$$

SOLUTION

Both powers are even. Use $\sin x\cos x = \dfrac{1}{2}\sin 2x$, then reduce:

$$\sin^2 x\cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{4}\cdot\frac{1 - \cos 4x}{2} = \frac{1}{8} - \frac{1}{8}\cos 4x.$$

Free sample from math.wiki for classroom use. Do not post. math.wiki

So

$$\int_0^{\pi/6} \sin^2 x \cos^2 x \, dx = \left[\frac{x}{8} - \frac{\sin 4x}{32}\right]_0^{\pi/6} = \frac{\pi}{48} - \frac{1}{32}\sin\frac{2\pi}{3} = \frac{\pi}{48} - \frac{\sqrt{3}}{64}.$$

The value is about 0.0384, positive as it must be.

SCORING

1 point for rewriting the integrand with first powers only, $\frac{1}{8} - \frac{1}{8}\cos 4x$ (or an equivalent route by power reducing); 1 point for the antiderivative $\frac{x}{8} - \frac{\sin 4x}{32}$; 1 point for the exact value $\frac{\pi}{48} - \frac{\sqrt{3}}{64}$.

8. Find $\int \dfrac{x^2}{\left(16 - x^2\right)^{3/2}}\, dx$. (3 points)

ANSWER

$$\frac{x}{\sqrt{16 - x^2}} - \arcsin\frac{x}{4} + C$$

SOLUTION

Let $x = 4\sin\theta$ with $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Then $dx = 4\cos\theta\, d\theta$ and $16 - x^2 = 16\cos^2\theta$, whose $\frac{3}{2}$ power is $64\cos^3\theta$ because $\cos\theta > 0$:

$$\int \frac{16\sin^2\theta \cdot 4\cos\theta}{64\cos^3\theta}\, d\theta = \int \tan^2\theta\, d\theta = \int \left(\sec^2\theta - 1\right) d\theta = \tan\theta - \theta + C.$$

The triangle has hypotenuse 4, opposite side x and adjacent side $\sqrt{16 - x^2}$, so $\tan\theta = \dfrac{x}{\sqrt{16 - x^2}}$ and $\theta = \arcsin\dfrac{x}{4}$:

$$\int \frac{x^2}{\left(16 - x^2\right)^{3/2}}\, dx = \frac{x}{\sqrt{16 - x^2}} - \arcsin\frac{x}{4} + C.$$

SCORING

1 point for $x = 4\sin\theta$ with the correct dx and $\left(16 - x^2\right)^{3/2} = 64\cos^3\theta$; 1 point for reducing to $\int \tan^2\theta\, d\theta = \tan\theta - \theta + C$; 1 point for the answer in x.

Free sample from math.wiki for classroom use. Do not post. math.wiki