Calculus 2 Unit 1 quiz 2: Integration by partial fractions, Choosing an integration technique (answer key)

7 questions, 20 points

ANSWERS AT A GLANCE

1. $2x - \tan x + C$
2. $F(x) = 5\ln(x+1) - 4\ln(x-3) - 5\ln 5$ for $x > 3$
3. $3\ln 3 + 2\ln 2 - \dfrac{3}{2}$
4. $\dfrac{x^2}{2} + 2x + 3\ln|x-1| - \dfrac{1}{x-1} + C$
5. $\dfrac{5}{2}\ln 2 - \dfrac{\pi}{4}$
6. $\sin(2\sqrt{x}) - \dfrac{1}{3}\sin^3(2\sqrt{x}) + C$
7. $11 - 6\ln\dfrac{3}{2}$

WORKED SOLUTIONS

1. Find $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx$. (2 points)

ANSWER

$2x - \tan x + C$

SOLUTION

Rewrite first with $\cos 2x = 2\cos^2 x - 1$: wherever $\cos x \neq 0$,

$$\frac{\cos 2x}{\cos^2 x} = 2 - \sec^2 x,$$

so $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx = 2x - \tan x + C$ on each interval where $\cos x \neq 0$. Check: $\dfrac{d}{dx}(2x - \tan x) = 2 - \sec^2 x$.

SCORING

1 point for rewriting the integrand as $2 - \sec^2 x$ (or an equal form such as $1 - \tan^2 x$); 1 point for $2x - \tan x + C$.

2. Find the function F with $F'(x) = \dfrac{x - 19}{x^2 - 2x - 3}$ for $x > 3$ and $F(4) = 0$. (3 points)

ANSWER

$F(x) = 5\ln(x+1) - 4\ln(x-3) - 5\ln 5$ for $x > 3$

SOLUTION

The bottom factors as $(x+1)(x-3)$, distinct linear factors, and the top has degree $1 < 2$. Cover-up: at $x = -1$, covering $x + 1$ leaves $\dfrac{x-19}{x-3} = \dfrac{-20}{-4} = 5$; at $x = 3$, covering $x - 3$ leaves $\dfrac{x-19}{x+1} = \dfrac{-16}{4} = -4$.

For $x > 3$ both factors are positive, so the absolute values can be dropped and

$$F(x) = 5\ln(x+1) - 4\ln(x-3) + C.$$

At $x = 4$: $5\ln 5 - 4\ln 1 + C = 0$, so $C = -5\ln 5$, and $F(x) = 5\ln(x+1) - 4\ln(x-3) - 5\ln 5$ for $x > 3$.

SCORING

1 point for the factoring and both constants 5 and −4; 1 point for the antiderivative on $x > 3$; 1 point for $C = -5\ln 5$ from $F(4) = 0$.

3. Evaluate $\displaystyle\int_1^3 \frac{5x^2 + 2x + 3}{x^3 + 2x^2 + x}\,dx$ exactly. (3 points)

ANSWER

Free sample from math.wiki for classroom use. Do not post. math.wiki

$3\ln 3 + 2\ln 2 - \dfrac{3}{2}$

SOLUTION

$x^3 + 2x^2 + x = x\left(x^2 + 2x + 1\right) = x(x+1)^2$, so

$$\frac{5x^2 + 2x + 3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing denominators: $5x^2 + 2x + 3 = A(x+1)^2 + Bx(x+1) + Cx$. At $x = 0$: $3 = A$. At $x = -1$: $5 - 2 + 3 = -C$, so $C = -6$. Matching x^2: $5 = A + B$, so $B = 2$. Then $\int \dfrac{-6}{(x+1)^2}\,dx = \dfrac{6}{x+1}$, so an antiderivative is

$$F(x) = 3\ln|x| + 2\ln|x+1| + \frac{6}{x+1}.$$

The bottom is zero only at 0 and -1, outside $[1, 3]$, so

$$F(3) - F(1) = \left(3\ln 3 + 2\ln 4 + \frac{3}{2}\right) - (0 + 2\ln 2 + 3) = 3\ln 3 + 2\ln 2 - \frac{3}{2} \approx 3.182.$$

SCORING

1 point for factoring the bottom as $x(x+1)^2$ and writing all three pieces (a form with no $\dfrac{B}{x+1}$ earns 0 here and for the next point); 1 point for $A = 3$, $B = 2$, $C = -6$; 1 point for the antiderivative, with $\dfrac{6}{x+1}$ from the power rule, and the value $3\ln 3 + 2\ln 2 - \dfrac{3}{2}$.

4. Find $\displaystyle\int \frac{x^3}{x^2 - 2x + 1}\,dx$. (3 points)

ANSWER

$\dfrac{x^2}{2} + 2x + 3\ln|x-1| - \dfrac{1}{x-1} + C$

SOLUTION

The top has degree 3 and the bottom degree 2, so divide: $(x+2)\left(x^2 - 2x + 1\right) = x^3 - 3x + 2$, so $x^3 = (x+2)\left(x^2 - 2x + 1\right) + 3x - 2$ and

$$\frac{x^3}{x^2 - 2x + 1} = x + 2 + \frac{3x - 2}{(x-1)^2}.$$

The remainder has a repeated linear factor: $\dfrac{3x-2}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$ with $3x - 2 = A(x-1) + B$. Matching x: $A = 3$; at $x = 1$: $B = 1$. So

$$\int \frac{x^3}{x^2 - 2x + 1}\,dx = \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C,$$

on $x < 1$ and on $x > 1$.

SCORING

Free sample from math.wiki for classroom use. Do not post. math.wiki

1 point for the quotient $x+2$ and remainder $3x-2$; 1 point for $\dfrac{3}{x-1}+\dfrac{1}{(x-1)^2}$; 1 point for the antiderivative with $-\dfrac{1}{x-1}$ and $+C$ (decomposing without dividing first earns 0 for the last two points).

5. Evaluate $\displaystyle\int_0^2 \frac{3x^2+4}{x^3+2x^2+4x+8}\,dx$ exactly. (3 points)

ANSWER

$\dfrac{5}{2}\ln 2-\dfrac{\pi}{4}$

SOLUTION

Group the bottom: $x^3+2x^2+4x+8=x^2(x+2)+4(x+2)=(x+2)\left(x^2+4\right)$, and x^2+4 is irreducible. So

$$\frac{3x^2+4}{(x+2)\left(x^2+4\right)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+4}.$$

Cover-up at $x=-2$: $A=\dfrac{16}{8}=2$. Then $3x^2+4=2\left(x^2+4\right)+(Bx+C)(x+2)$. Matching x^2: $B=1$. Matching the constant: $4=8+2C$, so $C=-2$. An antiderivative is

$$F(x)=2\ln|x+2|+\frac{1}{2}\ln\left(x^2+4\right)-\arctan\frac{x}{2}.$$

The bottom's only real zero is -2, outside $[0,2]$, so

$$F(2)-F(0)=\left(2\ln 4+\frac{1}{2}\ln 8-\frac{\pi}{4}\right)-\left(2\ln 2+\frac{1}{2}\ln 4\right)=\frac{11}{2}\ln 2-\frac{\pi}{4}-3\ln 2=\frac{5}{2}\ln 2-\frac{\pi}{4}\approx 0.948.$$

SCORING

1 point for factoring the bottom and finding $A=2$, $B=1$, $C=-2$; 1 point for the antiderivative, with $\dfrac{1}{2}\ln\left(x^2+4\right)$ and $-\arctan\dfrac{x}{2}$; 1 point for $\dfrac{5}{2}\ln 2-\dfrac{\pi}{4}$.

6. Find $\displaystyle\int \frac{\cos^3\left(2\sqrt{x}\right)}{\sqrt{x}}\,dx$ for $x>0$. (3 points)

ANSWER

$\sin\left(2\sqrt{x}\right)-\dfrac{1}{3}\sin^3\left(2\sqrt{x}\right)+C$

SOLUTION

The factor $\dfrac{1}{\sqrt{x}}$ is a constant times the derivative of the inside. Let $u=2\sqrt{x}$, so $du=\dfrac{dx}{\sqrt{x}}$:

$$\int \frac{\cos^3\left(2\sqrt{x}\right)}{\sqrt{x}}\,dx=\int \cos^3 u\,du.$$

An odd power of cosine: save one factor and use $\cos^2 u=1-\sin^2 u$, then $w=\sin u$, $dw=\cos u\,du$:

$$\int \left(1-\sin^2 u\right)\cos u\,du=\int\left(1-w^2\right)dw=w-\frac{w^3}{3}+C.$$

So the integral is $\sin\left(2\sqrt{x}\right)-\dfrac{1}{3}\sin^3\left(2\sqrt{x}\right)+C$.

SCORING

Free sample from math.wiki for classroom use. Do not post. math.wiki

1 point for $u = 2\sqrt{x}$ with $du = \dfrac{dx}{\sqrt{x}}$, reaching $\int \cos^3 u \, du$; 1 point for the odd-power rewrite $\left(1 - \sin^2 u\right)\cos u$; 1 point for the final antiderivative in x with $+C$.

7. Evaluate $\displaystyle\int_1^{64} \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ exactly. Choose one substitution that clears both roots. (3 points)

ANSWER

$11 - 6\ln\dfrac{3}{2}$

SOLUTION

Both roots are powers of $x^{1/6}$, so let $t = x^{1/6}$ for $x > 0$: then $x = t^6$, $dx = 6t^5\,dt$, $\sqrt{x} = t^3$ and $\sqrt[3]{x} = t^2$. The map is one-to-one for $t > 0$; $x = 1$ gives $t = 1$ and $x = 64$ gives $t = 2$. So

$$\int_1^{64} \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \int_1^{2} \frac{6t^5\,dt}{t^3 + t^2} = 6\int_1^{2} \frac{t^3}{t+1}\,dt.$$

The fraction is top-heavy. Dividing, $t^3 = (t+1)\left(t^2 - t + 1\right) - 1$, so $\dfrac{t^3}{t+1} = t^2 - t + 1 - \dfrac{1}{t+1}$ and

$$\begin{aligned} 6\int_1^{2} \frac{t^3}{t+1}\,dt &= \Big[2t^3 - 3t^2 + 6t - 6\ln(t+1)\Big]_1^{2} \\ &= (16 - 6\ln 3) - (5 - 6\ln 2) = 11 - 6\ln\frac{3}{2}. \end{aligned}$$

SCORING

1 point for $t = x^{1/6}$ with $dx = 6t^5\,dt$ and new limits 1 and 2, reaching $6\int_1^2 \dfrac{t^3}{t+1}\,dt$; 1 point for dividing to $t^2 - t + 1 - \dfrac{1}{t+1}$; 1 point for $11 - 6\ln\dfrac{3}{2}$.

Free sample from math.wiki for classroom use. Do not post. math.wiki