Calculus 2 Unit 1 test: Integration techniques (answer key)

12 questions, 51 points

ANSWERS AT A GLANCE

1. $\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2$
2. (a) shown below (b) $I_0 = 1 - \frac{1}{e}$, $I_3 = 6 - \frac{16}{e}$ (c) $e^{-1} \le e^{-x} \le 1$ on $[0,1]$; $6 - \frac{16}{e} \approx 0.114$ lies between 0.092 and 0.25
3. $\frac{5\pi}{8} + \frac{23}{12}$, about 3.880
4. $4\sqrt{3} - 2\ln\left(2+\sqrt{3}\right)$
5. (a) $\frac{1}{4}\ln\left(x^4+2x^2+5\right) + C$ (b) $\frac{1}{4}\arctan\frac{x^2+1}{2} + C$
6. $\frac{1}{2}\arctan 2$
7. $\frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(1+\sqrt{2}\right)$, about 0.266
8. (a) $\frac{\pi}{2}$ (b) $\frac{2}{3} + \frac{\sqrt{3}}{4\pi} \approx 0.804$ (c) $\alpha = \frac{\pi}{2}$
9. (a) $12\ln\frac{5}{4}$ meters (about 2.68 meters) (b) $t = 2$ seconds
10. (a) $f(x) = \frac{2}{x} + \frac{a-2}{x-2} + \frac{2a}{(x-2)^2}$ (b) $a = 2$; $\int_3^4 f(x)\,dx = 2\ln\frac{4}{3} + 2$
11. (a) Her second step undoes the first. (b) $I = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$ (c) $\frac{1}{5}\left(e^{2\pi}+1\right)$
12. (a) $\int_5^{10} 2\sqrt{100-x^2}\,dx = \frac{100\pi}{3} - 25\sqrt{3}$ square meters. (b) Shown below.

WORKED SOLUTIONS

1. Evaluate $\displaystyle\int_0^1 x^2 \arctan x\,dx$ exactly. (4 points)

ANSWER

$\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2$

SOLUTION

The inverse tangent is u: $u = \arctan x$ and $dv = x^2\,dx$, so $du = \frac{1}{1+x^2}\,dx$ and $v = \frac{x^3}{3}$:

$$\int_0^1 x^2\arctan x\,dx = \left[\frac{x^3}{3}\arctan x\right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx = \frac{\pi}{12} - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx.$$

Long division gives $\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$, so

$$\int_0^1 \frac{x^3}{1+x^2}\,dx = \left[\frac{x^2}{2} - \frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{1}{2} - \frac{1}{2}\ln 2.$$

Therefore

$$\int_0^1 x^2\arctan x\,dx = \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2 \approx 0.211.$$

SCORING

1 point for $u = \arctan x$, $v = \frac{x^3}{3}$; 1 point for the bracket $\frac{\pi}{12}$; 1 point for dividing to $x - \frac{x}{1+x^2}$ and integrating; 1 point for $\frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2$.

2. For whole numbers $n \ge 0$, let $I_n = \displaystyle\int_0^1 x^n e^{-x}\,dx$. (4 points)

(a) Use integration by parts to show that $I_n = nI_{n-1} - \frac{1}{e}$ for $n \ge 1$.

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(b) Find I_0, and use the formula in (a) to find I_3 exactly.

(c) Without integrating, explain why $\dfrac{1}{4e} \le I_3 \le \dfrac{1}{4}$, and check your answer to (b) against it.

ANSWER

(a) shown below (b) $I_0 = 1 - \dfrac{1}{e}$, $I_3 = 6 - \dfrac{16}{e}$ (c) $e^{-1} \le e^{-x} \le 1$ on $[0,1]$; $6 - \dfrac{16}{e} \approx 0.114$ lies between 0.092 and 0.25

SOLUTION

(a) Take $u = x^n$ and $dv = e^{-x}\,dx$, so $du = nx^{n-1}\,dx$ and $v = -e^{-x}$. For $n \ge 1$,

$$I_n = \Big[-x^n e^{-x} \Big]_0^1 + n \int_0^1 x^{n-1} e^{-x}\,dx = -\frac{1}{e} + 0 + nI_{n-1},$$

since $0^n = 0$ for $n \ge 1$. So $I_n = nI_{n-1} - \dfrac{1}{e}$.

(b) $I_0 = \displaystyle\int_0^1 e^{-x}\,dx = 1 - \frac{1}{e}$. Then

$$I_1 = 1 - \frac{1}{e} - \frac{1}{e} = 1 - \frac{2}{e}, \qquad I_2 = 2 - \frac{4}{e} - \frac{1}{e} = 2 - \frac{5}{e}, \qquad I_3 = 6 - \frac{15}{e} - \frac{1}{e} = 6 - \frac{16}{e}.$$

(c) On $[0,1]$, $e^{-1} \le e^{-x} \le 1$, so $\dfrac{1}{e}x^3 \le x^3 e^{-x} \le x^3$. Integrating over $[0,1]$, where $\int_0^1 x^3\,dx = \dfrac{1}{4}$, gives $\dfrac{1}{4e} \le I_3 \le \dfrac{1}{4}$. Since $\dfrac{1}{4e} \approx 0.092$ and $6 - \dfrac{16}{e} \approx 0.114$, the answer to (b) lies in the range. (It is near the low end because x^3 is largest near $x = 1$, where e^{-x} is near $\dfrac{1}{e}$.)

SCORING

1 point for the parts step with the bracket $[-x^n e^{-x}]_0^1 = -\dfrac{1}{e}$; 1 point for $I_0 = 1 - \dfrac{1}{e}$; 1 point for $I_3 = 6 - \dfrac{16}{e}$ through I_1 and I_2; 1 point for the bounds from $e^{-1} \le e^{-x} \le 1$ and the check.

3. Evaluate $\int_0^{\pi/4} (2\cos x + \cos 3x)^2\,dx$ exactly. (4 points)

ANSWER

$\dfrac{5\pi}{8} + \dfrac{23}{12}$, about 3.880

SOLUTION

Expand: $(2\cos x + \cos 3x)^2 = 4\cos^2 x + 4\cos x \cos 3x + \cos^2 3x$. Take the three terms one at a time.

Squares, by power reducing:

$$\int_0^{\pi/4} 4\cos^2 x\,dx = 4\left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/4} = 4\left(\frac{\pi}{8} + \frac{1}{4}\right) = \frac{\pi}{2} + 1,$$

$$\int_0^{\pi/4} \cos^2 3x\,dx = \left[\frac{x}{2} + \frac{\sin 6x}{12}\right]_0^{\pi/4} = \frac{\pi}{8} + \frac{1}{12}\sin\frac{3\pi}{2} = \frac{\pi}{8} - \frac{1}{12}.$$

The cross term has different frequencies, so use $\cos A \cos B = \dfrac{1}{2}\big[\cos(A-B) + \cos(A+B)\big]$:

$4\cos x \cos 3x = 2\cos 2x + 2\cos 4x$, and

$$\int_0^{\pi/4} (2\cos 2x + 2\cos 4x)\,dx = \left[\sin 2x + \frac{\sin 4x}{2}\right]_0^{\pi/4} = 1 + 0 = 1.$$

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Over this quarter turn the cross term does not vanish. The total is

$$\frac{\pi}{2}+1+\frac{\pi}{8}-\frac{1}{12}+1=\frac{5\pi}{8}+\frac{23}{12}\approx 3.880.$$

SCORING

1 point for expanding the square and turning the cross term into $2\cos 2x+2\cos 4x$ by a product to sum formula; 1 point for $\frac{\pi}{2}+1$ from $4\cos^2 x$; 1 point for $\frac{\pi}{8}-\frac{1}{12}$ from $\cos^2 3x$; 1 point for the cross term 1 and the total $\frac{5\pi}{8}+\frac{23}{12}$.

4. Evaluate $\int_1^{1+2\sqrt{3}} \frac{(x-1)^2}{\sqrt{x^2-2x+5}}\,dx$. You may use $\int \sec^3\theta\,d\theta=\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|+C$. (4 points)

ANSWER

$4\sqrt{3}-2\ln\left(2+\sqrt{3}\right)$

SOLUTION

Complete the square: $x^2-2x+5=(x-1)^2+4$. Let $x-1=2\tan\theta$, so $dx=2\sec^2\theta\,d\theta$ and $\sqrt{(x-1)^2+4}=2\sec\theta$ (as $\sec\theta>0$). The limits: $x=1$ gives $\tan\theta=0$, so $\theta=0$; $x=1+2\sqrt{3}$ gives $\tan\theta=\sqrt{3}$, so $\theta=\frac{\pi}{3}$.

$$\int_1^{1+2\sqrt{3}} \frac{(x-1)^2}{\sqrt{x^2-2x+5}}\,dx=\int_0^{\pi/3}\frac{4\tan^2\theta\cdot 2\sec^2\theta}{2\sec\theta}\,d\theta=4\int_0^{\pi/3}\tan^2\theta\sec\theta\,d\theta=4\int_0^{\pi/3}\left(\sec^3\theta-\sec\theta\right)d\theta.$$

With the given formula and $\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C$, an antiderivative of $\sec^3\theta-\sec\theta$ is $\frac{1}{2}\sec\theta\tan\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|$. At $\theta=\frac{\pi}{3}$, $\sec\theta=2$ and $\tan\theta=\sqrt{3}$; at $\theta=0$ it is $0-\frac{1}{2}\ln 1=0$. So

$$4\left(\sqrt{3}-\frac{1}{2}\ln\left(2+\sqrt{3}\right)\right)=4\sqrt{3}-2\ln\left(2+\sqrt{3}\right)\approx 4.294.$$

SCORING

1 point for completing the square and the substitution $x-1=2\tan\theta$ with dx and the root; 1 point for the new limits 0 and $\frac{\pi}{3}$; 1 point for rewriting $\tan^2\theta\sec\theta=\sec^3\theta-\sec\theta$ and integrating; 1 point for the exact value.

5. The two integrals have the same denominator but need different methods. Find (a) $\int\frac{x^3+x}{x^4+2x^2+5}\,dx$ and (b) $\int\frac{x}{x^4+2x^2+5}\,dx$. (4 points)

ANSWER

(a) $\frac{1}{4}\ln\left(x^4+2x^2+5\right)+C$ (b) $\frac{1}{4}\arctan\frac{x^2+1}{2}+C$

SOLUTION

(a) The derivative of the denominator is $4x^3+4x=4\left(x^3+x\right)$, a constant times the numerator. With $u=x^4+2x^2+5$, $du=4\left(x^3+x\right)dx$:

$$\int\frac{x^3+x}{x^4+2x^2+5}\,dx=\frac{1}{4}\int\frac{du}{u}=\frac{1}{4}\ln\left(x^4+2x^2+5\right)+C,$$

with no absolute value since $x^4+2x^2+5>0$. Partial fractions would have meant factoring a quartic.

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(b) Now the numerator is only x, but $x\,dx$ is half of $d\left(x^2\right)$. Let $w = x^2$, $dw = 2x\,dx$, and complete the square, $w^2 + 2w + 5 = (w+1)^2 + 4$:

$$\begin{aligned}\int \frac{x}{x^4 + 2x^2 + 5}\,dx &= \frac{1}{2}\int \frac{dw}{(w+1)^2 + 4} \\ &= \frac{1}{2}\cdot\frac{1}{2}\arctan\frac{w+1}{2} + C \\ &= \frac{1}{4}\arctan\frac{x^2+1}{2} + C.\end{aligned}$$

Check (b): the derivative is $\dfrac{1}{4}\cdot\dfrac{x}{1+\frac{(x^2+1)^2}{4}} = \dfrac{x}{4+(x^2+1)^2} = \dfrac{x}{x^4+2x^2+5}$.

SCORING

(a) 1 point for seeing the numerator as a constant times the derivative of the denominator; 1 point for $\dfrac{1}{4}\ln\left(x^4+2x^2+5\right) + C$. (b) 1 point for $w = x^2$ with the square completed, $(w+1)^2+4$; 1 point for $\dfrac{1}{4}\arctan\dfrac{x^2+1}{2} + C$.

6. Evaluate $\displaystyle\int_0^{\pi/4} \frac{dx}{\cos^2 x + 4\sin^2 x}$. (4 points)

ANSWER

$\dfrac{1}{2}\arctan 2$

SOLUTION

No factor is the derivative of the denominator as it stands, so rewrite first. On $\left[0, \dfrac{\pi}{4}\right]$, $\cos x \neq 0$; dividing top and bottom by $\cos^2 x$,

$$\frac{1}{\cos^2 x + 4\sin^2 x} = \frac{\sec^2 x}{1 + 4\tan^2 x}.$$

Now $\sec^2 x\,dx$ is the derivative of $\tan x$. Let $u = \tan x$, $du = \sec^2 x\,dx$; $x = 0$ gives $u = 0$ and $x = \dfrac{\pi}{4}$ gives $u = 1$:

$$\begin{aligned}\int_0^{\pi/4} \frac{dx}{\cos^2 x + 4\sin^2 x} &= \int_0^1 \frac{du}{1+4u^2} \\ &= \frac{1}{2}\arctan(2u)\Big|_0^1 = \frac{1}{2}\arctan 2.\end{aligned}$$

SCORING

1 point for dividing by $\cos^2 x$ to get $\dfrac{\sec^2 x}{1+4\tan^2 x}$; 1 point for $u = \tan x$ with limits 0 and 1; 1 point for the antiderivative $\dfrac{1}{2}\arctan(2u)$; 1 point for $\dfrac{1}{2}\arctan 2$.

7. Evaluate $\int_0^{\pi/4} \tan^2 x \sec x\,dx$ exactly. Show the integration by parts that the odd power of secant needs. (5 points)

ANSWER

$\dfrac{\sqrt{2}}{2} - \dfrac{1}{2}\ln\left(1+\sqrt{2}\right)$, about 0.266

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SOLUTION

The power of tangent is even and the power of secant is odd, so neither move of the lesson applies. Rewrite with $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \sec x \, dx = \int \sec^3 x \, dx - \int \sec x \, dx.$$

For $\int \sec^3 x \, dx$, use parts with $u = \sec x$, $dv = \sec^2 x \, dx$, so $du = \sec x \tan x \, dx$ and $v = \tan x$:

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx.$$

The integral comes back; solve for it:

$$\int \sec^3 x \, dx = \frac{1}{2}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + C.$$

So

$$\int \tan^2 x \sec x \, dx = \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C.$$

At $x = \dfrac{\pi}{4}$: $\sec x = \sqrt{2}$, $\tan x = 1$. At $x = 0$: $\sec x = 1$, $\tan x = 0$, and both terms are 0. So

$$\int_0^{\pi/4} \tan^2 x \sec x \, dx = \frac{\sqrt{2}}{2} - \frac{1}{2}\ln\left(1 + \sqrt{2}\right) \approx 0.266.$$

(Parts straight on $\int \tan^2 x \sec x \, dx$, with $u = \tan x$ and $dv = \sec x \tan x \, dx$, gives the same antiderivative.)

SCORING

1 point for rewriting as $\int \left(\sec^3 x - \sec x\right) dx$ (or setting up parts directly on $\tan x \cdot \sec x \tan x$); 2 points for integration by parts on the secant cube: 1 for u, dv, du, v and the formula, 1 for solving the returning integral; 1 point for $\int \sec x \, dx = \ln|\sec x + \tan x|$; 1 point for the exact value $\dfrac{\sqrt{2}}{2} - \dfrac{1}{2}\ln\left(1 + \sqrt{2}\right)$. Quoting the formula for $\int \sec^3 x \, dx$ without the parts work earns neither of the 2 parts points.

8. Yuna's reading lamp has a leading-edge dimmer. In a made-up model, over each half cycle the line voltage is $V = 170 \sin\theta$ volts for $0 \le \theta \le \pi$, where θ is the phase angle. The dimmer blocks the voltage for $0 \le \theta < \alpha$ and passes it for $\alpha \le \theta \le \pi$. The lamp's power at each instant is proportional to V^2, so the share of full power that reaches the lamp is

$$\frac{\int_\alpha^\pi \sin^2\theta \, d\theta}{\int_0^\pi \sin^2\theta \, d\theta}.$$

(4 points)

(a) Find $\int_0^\pi \sin^2\theta \, d\theta$.

(b) Find the share when $\alpha = \dfrac{\pi}{3}$, exactly, and then use a calculator and round to three decimal places.

(c) Find the switch-on angle α that gives exactly half of full power.

ANSWER

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{2}{3} + \dfrac{\sqrt{3}}{4\pi} \approx 0.804$ (c) $\alpha = \dfrac{\pi}{2}$

SOLUTION

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The power is even, so reduce it: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, with antiderivative $\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}$.

(a) $\left[\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}\right]_0^{\pi} = \dfrac{\pi}{2}$.

(b) $\int_{\pi/3}^{\pi} \sin^2\theta\, d\theta = \dfrac{\pi}{2} - \left(\dfrac{\pi}{6} - \dfrac{1}{4}\sin\dfrac{2\pi}{3}\right) = \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{8}$. The share is

$$\frac{\frac{\pi}{3} + \frac{\sqrt{3}}{8}}{\frac{\pi}{2}} = \frac{2}{3} + \frac{\sqrt{3}}{4\pi} \approx 0.804.$$

Blocking the first third of the half cycle removes only about 20 percent of the power, because $\sin^2\theta$ is small near $\theta = 0$.

(c) The share as a function of α is $\dfrac{2}{\pi}\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2} + \dfrac{\sin 2\alpha}{4}\right) = 1 - \dfrac{\alpha}{\pi} + \dfrac{\sin 2\alpha}{2\pi}$. Setting it equal to $\dfrac{1}{2}$: at $\alpha = \dfrac{\pi}{2}$, $\sin 2\alpha = \sin\pi = 0$ and the share is $1 - \dfrac{1}{2} = \dfrac{1}{2}$. This is the only solution, since the share decreases as α grows (its derivative is $-\dfrac{\sin^2\alpha}{\pi/2} \le 0$, zero only at the ends). It also follows from symmetry: $\sin^2\theta$ is symmetric about $\theta = \dfrac{\pi}{2}$, so each half of $[0,\pi]$ carries half the integral.

SCORING

1 point for (a) $\dfrac{\pi}{2}$ by power reducing; 2 points for (b): 1 for $\int_{\pi/3}^{\pi}\sin^2\theta\,d\theta = \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{8}$, 1 for the share $\dfrac{2}{3} + \dfrac{\sqrt{3}}{4\pi} \approx 0.804$; 1 point for (c) $\alpha = \dfrac{\pi}{2}$ with a reason (the computed share or the symmetry of $\sin^2\theta$ about $\dfrac{\pi}{2}$).

9. In a distribution center, Andre tests an automated pallet cart as it eases into a loading bay. In a made-up model its velocity is

$$v(t) = \frac{12}{t^2 + 5t + 6}$$

meters per second, t seconds after the test starts. (4 points)

(a) Find the exact distance the cart travels in the first 3 seconds.

(b) Find the exact time at which the cart has traveled $12\ln\dfrac{6}{5}$ meters.

ANSWER

(a) $12\ln\dfrac{5}{4}$ meters (about 2.68 meters) (b) $t = 2$ seconds

SOLUTION

$v(t) > 0$ for $t \ge 0$, so distance is $\int v(t)\,dt$. The bottom is $(t+2)(t+3)$. Cover-up: at $t=-2$, $\dfrac{12}{1} = 12$; at $t=-3$, $\dfrac{12}{-1} = -12$. So $v(t) = \dfrac{12}{t+2} - \dfrac{12}{t+3}$, and both factors are positive for $t \ge 0$, so the distance in the first T seconds is

$$\int_0^T v(t)\,dt = \left[12\ln\frac{t+2}{t+3}\right]_0^T = 12\ln\frac{T+2}{T+3} - 12\ln\frac{2}{3} = 12\ln\frac{3(T+2)}{2(T+3)}.$$

(a) $T = 3$: $12\ln\dfrac{15}{12} = 12\ln\dfrac{5}{4} \approx 2.68$ meters.

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(b) Set $12\ln\dfrac{3(T+2)}{2(T+3)} = 12\ln\dfrac{6}{5}$. Since ln is one-to-one, $\dfrac{3(T+2)}{2(T+3)} = \dfrac{6}{5}$, so $15T+30 = 12T+36$ and $T = 2$ seconds. (This is before 3 seconds, as it should be, since $\dfrac{6}{5} < \dfrac{5}{4}$.)

SCORING

1 point for $v(t) = \dfrac{12}{t+2} - \dfrac{12}{t+3}$; 1 point for the antiderivative $12\ln\dfrac{t+2}{t+3}$ (or the two logs separately); 1 point for $12\ln\dfrac{5}{4}$ meters; 1 point for setting the distance equal to $12\ln\dfrac{6}{5}$ and solving to $t = 2$ seconds.

10. For a constant a, let

$$f(x) = \frac{ax^2 - 4x + 8}{x(x-2)^2}.$$

(4 points)

(a) Write the partial fraction form of f, and find each constant in terms of a.

(b) Find the value of a for which an antiderivative of f contains no term $\ln|x-2|$. For that a, evaluate $\displaystyle\int_3^4 f(x)\,dx$ exactly.

ANSWER

(a) $f(x) = \dfrac{2}{x} + \dfrac{a-2}{x-2} + \dfrac{2a}{(x-2)^2}$ (b) $a = 2$; $\int_3^4 f(x)\,dx = 2\ln\dfrac{4}{3} + 2$

SOLUTION

(a) The form is $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$, and clearing denominators gives

$$ax^2 - 4x + 8 = A(x-2)^2 + Bx(x-2) + Cx.$$

At $x = 0$: $8 = 4A$, so $A = 2$. At $x = 2$: $4a - 8 + 8 = 2C$, so $C = 2a$. Matching x^2: $a = A + B$, so $B = a - 2$.

(b) Only the piece $\dfrac{B}{x-2}$ integrates to a logarithm of $|x-2|$; the piece over $(x-2)^2$ gives $-\dfrac{C}{x-2}$. So the log term is absent exactly when $B = a - 2 = 0$, that is $a = 2$. Then $C = 4$ and $f(x) = \dfrac{2}{x} + \dfrac{4}{(x-2)^2}$ (check: $2(x-2)^2 + 4x = 2x^2 - 4x + 8$). An antiderivative is $F(x) = 2\ln|x| - \dfrac{4}{x-2}$, and $[3,4]$ contains neither 0 nor 2, so

$$F(4) - F(3) = (2\ln 4 - 2) - (2\ln 3 - 4) = 2\ln\frac{4}{3} + 2 \approx 2.575.$$

SCORING

1 point for the form with all three pieces; 1 point for $A = 2$, $B = a - 2$, $C = 2a$; 1 point for $a = 2$ with the reason that B must be 0; 1 point for $2\ln\dfrac{4}{3} + 2$.

11. Jenny wants $I = \displaystyle\int e^{2x}\sin x\,dx$. Her first step takes $u = \sin x$ and $dv = e^{2x}\,dx$:

$$I = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx.$$

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For the new integral she takes $u = e^{2x}$ and $dv = \cos x\,dx$:

$$\int e^{2x}\cos x\,dx = e^{2x}\sin x - 2I.$$

Substituting, she gets $I = I$. (5 points)

(a) Her algebra has no slip. Explain why her second step learned nothing.

(b) Find I correctly.

(c) Evaluate $\displaystyle\int_0^{\pi} e^{2x}\sin x\,dx$ exactly.

ANSWER

(a) Her second step undoes the first. (b) $I = \dfrac{e^{2x}}{5}\left(2\sin x - \cos x\right) + C$ (c) $\dfrac{1}{5}\left(e^{2\pi}+1\right)$

SOLUTION

(a) In the first step the exponential was integrated and the sine differentiated. In the second step she swapped roles, differentiating the exponential and integrating the cosine back to $\sin x$. That reverses the first step exactly, so the equation returns as $I = A + I$ with $A = 0$, which is true and says nothing.

(b) Keep the exponential as dv: $u = \cos x$, $dv = e^{2x}\,dx$, so $du = -\sin x\,dx$ and $v = \dfrac{1}{2}e^{2x}$:

$$\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\cos x + \frac{1}{2}I.$$

Substituting into her first step, $I = \dfrac{1}{2}e^{2x}\sin x - \dfrac{1}{4}e^{2x}\cos x - \dfrac{1}{4}I$, so $\dfrac{5}{4}I = \dfrac{e^{2x}}{4}(2\sin x - \cos x)$ and

$$I = \frac{e^{2x}}{5}\left(2\sin x - \cos x\right) + C.$$

Check: the derivative is $\dfrac{2e^{2x}}{5}(2\sin x - \cos x) + \dfrac{e^{2x}}{5}(2\cos x + \sin x) = e^{2x}\sin x$.

(c) At π: $\dfrac{e^{2\pi}}{5}(0+1) = \dfrac{e^{2\pi}}{5}$. At 0: $\dfrac{1}{5}(0-1) = -\dfrac{1}{5}$. So the integral is $\dfrac{1}{5}\left(e^{2\pi}+1\right) \approx 107.3$, positive as it must be, since $\sin x \geq 0$ on $[0,\pi]$.

SCORING

(a) 1 point: says the second step reverses the first (the roles of the two factors are swapped back). (b) 1 point for the second step with $dv = e^{2x}\,dx$; 1 point for the equation $\dfrac{5}{4}I = \dfrac{e^{2x}}{4}(2\sin x - \cos x)$; 1 point for I with $+C$. (c) 1 point for $\dfrac{1}{5}\left(e^{2\pi}+1\right)$.

12. Jasmine is laying out a practice field for her club. A circle of radius 10 meters is painted around a center spot, and a straight line painted 5 meters from the spot cuts off part of the circle. The grass in the part of the disk beyond the line is to be replaced. (5 points)

(a) Put the spot at the origin and the line at $x = 5$. Write the area of the part of the disk beyond the line as a definite integral, and evaluate it exactly with a trigonometric substitution.

(b) Show that a line at distance d from the center of a circle of radius r, with $0 \leq d < r$, cuts off a piece of area $r^2 \arccos\dfrac{d}{r} - d\sqrt{r^2 - d^2}$.

ANSWER

(a) $\displaystyle\int_5^{10} 2\sqrt{100 - x^2}\,dx = \frac{100\pi}{3} - 25\sqrt{3}$ square meters. (b) Shown below.

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SOLUTION

(a) The circle is $x^2 + y^2 = 100$. At each x from 5 to 10 the disk runs from $y = -\sqrt{100 - x^2}$ to $y = \sqrt{100 - x^2}$, a height of $2\sqrt{100 - x^2}$, so the area is $\int_5^{10} 2\sqrt{100 - x^2}\,dx$.

Let $x = 10\sin\theta$ with $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, so $dx = 10\cos\theta\,d\theta$ and $\sqrt{100 - x^2} = 10\cos\theta$. The limits: $x = 5$ gives $\theta = \arcsin\dfrac{1}{2} = \dfrac{\pi}{6}$, and $x = 10$ gives $\theta = \dfrac{\pi}{2}$. Using $\int \cos^2\theta\,d\theta = \dfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$,

$$\int_{\pi/6}^{\pi/2} 2 \cdot 10\cos\theta \cdot 10\cos\theta\,d\theta = 100\Big[\theta + \sin\theta\cos\theta\Big]_{\pi/6}^{\pi/2} = 100\left(\frac{\pi}{2} - \frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) = \frac{100\pi}{3} - 25\sqrt{3}.$$

That is about 61.4 square meters.

(b) The same set-up gives $\int_d^r 2\sqrt{r^2 - x^2}\,dx$. Let $x = r\sin\theta$; the limits become $\alpha = \arcsin\dfrac{d}{r}$ and $\dfrac{\pi}{2}$, and

$$\int_{\alpha}^{\pi/2} 2r^2\cos^2\theta\,d\theta = r^2\Big[\theta + \sin\theta\cos\theta\Big]_{\alpha}^{\pi/2} = r^2\left(\frac{\pi}{2} - \alpha - \sin\alpha\cos\alpha\right).$$

Here $\sin\alpha = \dfrac{d}{r}$ and $\cos\alpha = \dfrac{\sqrt{r^2 - d^2}}{r}$ (positive, as $0 \le \alpha < \dfrac{\pi}{2}$), and $\dfrac{\pi}{2} - \arcsin\dfrac{d}{r} = \arccos\dfrac{d}{r}$. So the area is $r^2\arccos\dfrac{d}{r} - d\sqrt{r^2 - d^2}$. With $r = 10$ and $d = 5$ this is $100\cdot\dfrac{\pi}{3} - 5\sqrt{75}$, the answer to (a); with $d = 0$ it is $\dfrac{\pi r^2}{2}$, half the disk.

SCORING

(a) 1 point for the integral $\int_5^{10} 2\sqrt{100 - x^2}\,dx$; 1 point for $x = 10\sin\theta$ with the limits $\dfrac{\pi}{6}$ and $\dfrac{\pi}{2}$; 1 point for $\dfrac{100\pi}{3} - 25\sqrt{3}$. (b) 1 point for the general substitution with limits $\arcsin\dfrac{d}{r}$ and $\dfrac{\pi}{2}$; 1 point for evaluating and using $\dfrac{\pi}{2} - \arcsin\dfrac{d}{r} = \arccos\dfrac{d}{r}$ to reach the formula.

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