Calculus 2 Unit 1 worksheet: Choosing an integration technique (answer key)

ANSWERS AT A GLANCE

1. (a) $-2\cos x + C$ (b) $\sin x - \cos x + C$ (c) $e^x + x + C$
2. (a) $-\ln(1+e^{-x}) + C$ (b) $x - \ln(1+e^x) + C$ (c) The two are equal for every x, so the constant is 0.
3. $\frac{1}{5}\ln x - \frac{2}{5}\ln(5+\sqrt{x}) + C$
4. (a) $\ln x\,\ln(\ln x) - \ln x + C$ (b) $2e^{\sin x}(\sin x - 1) + C$
5. (a) $\frac{1}{3}\left(x^2+4\right)^{3/2} - 4\sqrt{x^2+4} + C$
   (b) $\frac{x}{2}\sqrt{x^2+4} - 2\ln\left(x+\sqrt{x^2+4}\right) + C$
   (c) In (a) the odd power gives $x^3\,dx = x^2\cdot x\,dx$, where $x\,dx$ is half of du for $u = x^2+4$ and $x^2 = u-4$; in (b) the even power leaves $x^2\,dx$, which cannot be written as a function of u times du without a new root, so the root needs $x = 2\tan\theta$.
6. $E(3) \approx 8.041$. It has a value because $\frac{e^t}{t}$ is continuous on $[1,3]$; only a formula for the antiderivative is missing.
7. (a) $R(x) = x$, so $\int \left(2x^2+1\right)e^{x^2}\,dx = xe^{x^2} + C$. (b) A pole of R of order k would give R' a pole of order $k+1$ that $2xR$ cannot cancel, but 1 has no pole; a nonzero polynomial R of degree n makes $R' + 2xR$ of degree $n+1 \geq 1$, and $R = 0$ gives 0. So no R exists, and $\int e^{x^2}\,dx$ is not elementary.
8. (a) F increases on $(0,\sqrt{\pi})$ and $\left(\sqrt{2\pi},3\right)$ and decreases on $\left(\sqrt{\pi},\sqrt{2\pi}\right)$; local maximum at $x = \sqrt{\pi}$, local minimum at $x = \sqrt{2\pi}$.
   (b) $x = \sqrt{\frac{\pi}{2}}$, $x = \sqrt{\frac{3\pi}{2}}$ and $x = \sqrt{\frac{5\pi}{2}}$.

WORKED SOLUTIONS

1. Each integrand becomes a basic form after a rewrite (cancel a factor or use an identity). Rewrite first, then integrate.

(a) $\displaystyle\int \frac{\sin 2x}{\cos x}\,dx$ (b) $\displaystyle\int \frac{\cos 2x}{\cos x - \sin x}\,dx$ (c) $\displaystyle\int \frac{e^{2x}-1}{e^x-1}\,dx$

ANSWER

(a) $-2\cos x + C$ (b) $\sin x - \cos x + C$ (c) $e^x + x + C$

SOLUTION

(a) $\sin 2x = 2\sin x\cos x$, so wherever $\cos x \neq 0$ the integrand is $2\sin x$, and $\int 2\sin x\,dx = -2\cos x + C$ on each interval where $\cos x \neq 0$.

(b) $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x)$, so wherever $\cos x \neq \sin x$ the integrand is $\cos x + \sin x$, and the integral is $\sin x - \cos x + C$ on each such interval.

(c) $e^{2x} - 1 = (e^x - 1)(e^x + 1)$, so for $x \neq 0$ the integrand is $e^x + 1$, and the integral is $e^x + x + C$ on $x < 0$ and on $x > 0$.

Check any of them by differentiating: for (b), $\frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x$.

2. Find $\displaystyle\int \frac{dx}{1+e^x}$ by two routes. (a) Multiply the top and bottom by e^{-x}, then substitute. (b) Substitute $u = e^x$, then use partial fractions. (c) Show that your two answers differ by a constant, and find that constant.

ANSWER

(a) $-\ln(1+e^{-x}) + C$ (b) $x - \ln(1+e^x) + C$ (c) The two are equal for every x, so the constant is 0.

SOLUTION

(a) $\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}+1}$. With $w = 1+e^{-x}$, $dw = -e^{-x}\,dx$, and $w > 0$:

$$\int \frac{e^{-x}}{1+e^{-x}}\,dx = -\int \frac{dw}{w} = -\ln\left(1+e^{-x}\right) + C.$$

(b) With $u = e^x$, $du = e^x\,dx = u\,dx$, so $dx = \frac{du}{u}$ and the integral is $\int \frac{du}{u(1+u)}$. By cover-up, $\frac{1}{u(1+u)} = \frac{1}{u} - \frac{1}{1+u}$, so, with $u > 0$,

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$$\int \frac{du}{u(1+u)} = \ln u - \ln(1+u) + C = x - \ln\left(1+e^{x}\right) + C.$$

(c) $-\ln\left(1+e^{-x}\right) = -\ln\dfrac{e^{x}+1}{e^{x}} = x - \ln\left(1+e^{x}\right)$. The two antiderivatives are the same function, so they differ by the constant 0. Route (a) used one rewrite and one substitution; route (b) needed a decomposition as well.

3. Find $\displaystyle\int \frac{dx}{x\left(5+\sqrt{x}\right)}$ for $x > 0$.

ANSWER

$\dfrac{1}{5}\ln x - \dfrac{2}{5}\ln\left(5+\sqrt{x}\right) + C$

SOLUTION

The root is the obstacle, so let $t = \sqrt{x}$ for $x > 0$: then $x = t^{2}$ and $dx = 2t\,dt$, and

$$\int \frac{dx}{x\left(5+\sqrt{x}\right)} = \int \frac{2t\,dt}{t^{2}\left(5+t\right)} = 2\int \frac{dt}{t\left(t+5\right)}.$$

This is a rational function with two distinct linear factors. By cover-up, $\dfrac{1}{t(t+5)} = \dfrac{1}{5}\left(\dfrac{1}{t} - \dfrac{1}{t+5}\right)$, so

$$2\int \frac{dt}{t\left(t+5\right)} = \frac{2}{5}\left(\ln t - \ln\left(t+5\right)\right) + C.$$

With $t = \sqrt{x}$ and $\dfrac{2}{5}\ln\sqrt{x} = \dfrac{1}{5}\ln x$, the integral is $\dfrac{1}{5}\ln x - \dfrac{2}{5}\ln\left(5+\sqrt{x}\right) + C$. A substitution set up partial fractions.

4. Each integral needs a substitution first and then integration by parts. Find (a) $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$ for $x > 1$; (b) $\displaystyle\int \sin 2x\, e^{\sin x}\,dx$.

ANSWER

(a) $\ln x\,\ln(\ln x) - \ln x + C$ (b) $2e^{\sin x}(\sin x - 1) + C$

SOLUTION

(a) The factor $\dfrac{1}{x}$ is the derivative of $\ln x$, so let $u = \ln x$, $du = \dfrac{dx}{x}$, with $u > 0$. The integral becomes $\int \ln u\,du$, a single logarithm, done by parts with $dv = du$: $\int \ln u\,du = u\ln u - u + C$. So

$$\int \frac{\ln(\ln x)}{x}\,dx = \ln x\,\ln(\ln x) - \ln x + C.$$

(b) Rewrite first: $\sin 2x = 2\sin x\cos x$, and $\cos x\,dx$ is the derivative of the exponent. With $u = \sin x$, $du = \cos x\,dx$:

$$\int \sin 2x\, e^{\sin x}\,dx = 2\int u\,e^{u}\,du.$$

A power times an exponential calls for parts with the power differentiated: $\int ue^{u}\,du = ue^{u} - e^{u} + C$. So the integral is $2e^{\sin x}(\sin x - 1) + C$.

Check (b): $\dfrac{d}{dx}\left[2e^{\sin x}(\sin x - 1)\right] = 2\cos x\, e^{\sin x}(\sin x - 1) + 2e^{\sin x}\cos x = 2\sin x\cos x\, e^{\sin x}$.

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5. Both integrals contain $\sqrt{x^2+4}$. Find (a) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$ and (b) $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$. (c) In one sentence, say why (a) needs no trigonometric substitution but (b) does.

ANSWER

(a) $\dfrac{1}{3}\left(x^2+4\right)^{3/2} - 4\sqrt{x^2+4} + C$

(b) $\dfrac{x}{2}\sqrt{x^2+4} - 2\ln\left(x+\sqrt{x^2+4}\right) + C$

(c) In (a) the odd power gives $x^3\,dx = x^2\cdot x\,dx$, where $x\,dx$ is half of du for $u = x^2+4$ and $x^2 = u-4$; in (b) the even power leaves $x^2\,dx$, which cannot be written as a function of u times du without a new root, so the root needs $x = 2\tan\theta$.

SOLUTION

(a) Let $u = x^2+4$, so $du = 2x\,dx$ and $x^2 = u-4$:

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+4}}\,dx &= \frac{1}{2}\int (u-4)\,u^{-1/2}\,du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 8u^{1/2}\right) + C \\ &= \frac{1}{3}\left(x^2+4\right)^{3/2} - 4\sqrt{x^2+4} + C.\end{aligned}$$

(b) Let $x = 2\tan\theta$ with $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, so $dx = 2\sec^2\theta\,d\theta$ and $\sqrt{x^2+4} = 2\sec\theta$. Then

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+4}}\,dx &= \int \frac{4\tan^2\theta\cdot 2\sec^2\theta}{2\sec\theta}\,d\theta \\ &= 4\int \left(\sec^3\theta - \sec\theta\right)d\theta.\end{aligned}$$

By parts, $\int \sec^3\theta\,d\theta = \dfrac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right) + C$, so the integral is $2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C$. From the triangle, $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$ and $\tan\theta = \dfrac{x}{2}$; the $-2\ln\dfrac{1}{2}$ joins the constant:

$$\int \frac{x^2}{\sqrt{x^2+4}}\,dx = \frac{x}{2}\sqrt{x^2+4} - 2\ln\left(x+\sqrt{x^2+4}\right) + C.$$

(c) As in the answer: one more factor of x is what lets $u = x^2+4$ consume the whole integrand.

6. The function $\dfrac{e^x}{x}$ has no elementary antiderivative. Let $E(x) = \displaystyle\int_1^x \frac{e^t}{t}\,dt$. Use a calculator and Simpson's rule with four subintervals to estimate $E(3)$, rounded to three decimal places, and say why $E(3)$ has a value at all.

ANSWER

$E(3) \approx 8.041$. It has a value because $\dfrac{e^t}{t}$ is continuous on $[1,3]$; only a formula for the antiderivative is missing.

SOLUTION

$h = \dfrac{3-1}{4} = \dfrac{1}{2}$, with nodes 1, 1.5, 2, 2.5, 3 and $f(t) = \dfrac{e^t}{t}$: $f(1) \approx 2.71828$, $f(1.5) \approx 2.98779$, $f(2) \approx 3.69453$, $f(2.5) \approx 4.87300$, $f(3) \approx 6.69518$. Simpson's rule gives

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$$E(3) \approx \frac{1/2}{3}\Big[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\Big] \approx \frac{48.24568}{6} \approx 8.041.$$

(A finer rule gives 8.039.) The integral has a value because the integrand is continuous on $[1, 3]$; only a formula for the antiderivative is missing.

7. Take $g(x) = x^2$ in the Liouville criterion, so $\displaystyle\int f(x)e^{x^2}\,dx$ is elementary exactly when some rational function R has $R' + 2xR = f$. (a) For $f(x) = 2x^2 + 1$, find such an R of the form $R(x) = ax + b$, and use it to find $\displaystyle\int \left(2x^2 + 1\right) e^{x^2}\,dx$. (b) For $f(x) = 1$, show that no rational R has $R' + 2xR = 1$, and conclude that $\displaystyle\int e^{x^2}\,dx$ is not elementary. Look first at a zero of the denominator of R, then at a polynomial R.

ANSWER

(a) $R(x) = x$, so $\int \left(2x^2 + 1\right) e^{x^2}\,dx = xe^{x^2} + C$. (b) A pole of R of order k would give R' a pole of order $k + 1$ that $2xR$ cannot cancel, but 1 has no pole; a nonzero polynomial R of degree n makes $R' + 2xR$ of degree $n + 1 \geq 1$, and $R = 0$ gives 0. So no R exists, and $\int e^{x^2}\,dx$ is not elementary.

SOLUTION

(a) With $R = ax + b$, $R' + 2xR = 2ax^2 + 2bx + a$. Matching $2x^2 + 1$ term by term: $2a = 2$, $2b = 0$ and $a = 1$. So $R(x) = x$, and the criterion gives

$$\int \left(2x^2 + 1\right) e^{x^2}\,dx = xe^{x^2} + C.$$

Check: $\dfrac{d}{dx}\left(xe^{x^2}\right) = e^{x^2} + 2x^2e^{x^2}$.

(b) Here $f = 1$, so the criterion asks for a rational R with $R' + 2xR = 1$.

A zero of the denominator. Suppose R has a pole of order $k \geq 1$ at some c. Then R' has a pole of order $k + 1$ at c, while $2xR$ has a pole of order at most k there, so it cannot cancel it. The left side then has a pole at c, but the right side, 1, has none. So R has no poles: it is a polynomial.

A polynomial. $R = 0$ gives $0 \neq 1$. If R is a nonzero polynomial of degree $n \geq 0$, then $2xR$ has degree $n + 1$ and R' has degree less than n, so $R' + 2xR$ has degree $n + 1 \geq 1$ and is not the constant 1.

No rational R works, so by the criterion $\int e^{x^2}\,dx$ is not elementary. Compare (a): there the extra $2x^2$ in f is exactly what the top term of $2xR$ can match.

8. Let $F(x) = \displaystyle\int_0^x \sin\left(t^2\right)\,dt$ for $0 \leq x \leq 3$; $\sin\left(x^2\right)$ has no elementary antiderivative. (a) Find the open intervals in (0, 3) on which F increases and on which it decreases, and the x-values of its local maximum and local minimum there. (b) Find the x-values of the inflection points of the graph of F in (0, 3). Give exact values.

ANSWER

(a) F increases on $(0, \sqrt{\pi})$ and $\left(\sqrt{2\pi}, 3\right)$ and decreases on $\left(\sqrt{\pi}, \sqrt{2\pi}\right)$; local maximum at $x = \sqrt{\pi}$, local minimum at $x = \sqrt{2\pi}$.

(b) $x = \sqrt{\frac{\pi}{2}}$, $x = \sqrt{\frac{3\pi}{2}}$ and $x = \sqrt{\frac{5\pi}{2}}$.

SOLUTION

(a) By the fundamental theorem, $F'(x) = \sin\left(x^2\right)$. For $0 < x < 3$, x^2 runs over (0, 9), and $9 < 3\pi$. So $\sin\left(x^2\right) > 0$ for $0 < x^2 < \pi$, < 0 for $\pi < x^2 < 2\pi$, and > 0 for $2\pi < x^2 < 9$. F increases on $(0, \sqrt{\pi})$ and

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$\left(\sqrt{2\pi}, 3\right)$ and decreases on $\left(\sqrt{\pi}, \sqrt{2\pi}\right)$, so it has a local maximum at $x = \sqrt{\pi}$ and a local minimum at $x = \sqrt{2\pi}$.

(b) $F''(x) = 2x \cos\left(x^2\right)$. On (0, 3), $2x > 0$, and $\cos\left(x^2\right)$ changes sign where $x^2 = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$ (all below 9; the next, $\dfrac{7\pi}{2} \approx 11.0$, is not). So F changes concavity at $x = \sqrt{\frac{\pi}{2}}$, $\sqrt{\frac{3\pi}{2}}$ and $\sqrt{\frac{5\pi}{2}}$. None of this needed a formula for F.

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