Calculus 2 Unit 1 worksheet: Integration by parts (answer key)

ANSWERS AT A GLANCE

1. (a) $2\sqrt{x}\ln x - 4\sqrt{x} + C$, for $x > 0$ (b) $x\arctan 3x - \frac{1}{6}\ln\left(1+9x^2\right) + C$, for all x
2. $-\frac{1}{2}(6x+7)\cos 2x + \frac{3}{2}\sin 2x + C$
3. (a) $-\frac{1}{2}$ (b) $\frac{5}{2}\ln 5 - 2$
4. $\frac{x^2(x-3)^7}{7} - \frac{x(x-3)^8}{28} + \frac{(x-3)^9}{252} + C$
5. (a) $I = -\frac{e^{-x}}{10}\left(\sin 3x + 3\cos 3x\right) + C$ (b) $\frac{3}{10}\left(1+e^{-\pi}\right)$
6. (a) $\frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1) + C$ (b) $\frac{x^2-1}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} + C$ (c) the two logarithm terms of (a) combine into the one of (b)
7. (a) $N(T) = 60 - 30e^{-T}\left(T^2+2T+2\right)$ (b) $60 - 780e^{-4}$, about 46 meteors (c) $150e^{-1} - 510e^{-3}$, about 30 meteors

WORKED SOLUTIONS

1. Find each integral by parts. Name u and dv each time, and say where the answer holds.

(a) $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx$ (b) $\displaystyle\int \arctan 3x\,dx$

ANSWER

(a) $2\sqrt{x}\ln x - 4\sqrt{x} + C$, for $x > 0$ (b) $x\arctan 3x - \frac{1}{6}\ln\left(1+9x^2\right) + C$, for all x

SOLUTION

(a) The logarithm is u: $u = \ln x$ and $dv = x^{-1/2}\,dx$, so $du = \frac{1}{x}\,dx$ and $v = 2\sqrt{x}$. For $x > 0$,

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - \int 2\sqrt{x}\cdot\frac{1}{x}\,dx = 2\sqrt{x}\ln x - \int 2x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C.$$

Check: the derivative is $\frac{\ln x}{\sqrt{x}} + \frac{2\sqrt{x}}{x} - \frac{2}{\sqrt{x}} = \frac{\ln x}{\sqrt{x}}$.

(b) A single factor, so $u = \arctan 3x$ and $dv = dx$: $du = \frac{3}{1+9x^2}\,dx$ and $v = x$. The new integral yields to $w = 1+9x^2$, $dw = 18x\,dx$, so $3x\,dx = \frac{1}{6}\,dw$:

$$\int \arctan 3x\,dx = x\arctan 3x - \int \frac{3x}{1+9x^2}\,dx = x\arctan 3x - \frac{1}{6}\ln\left(1+9x^2\right) + C.$$

No absolute value is needed since $1+9x^2 > 0$, and the answer holds for all x.

2. Find $\displaystyle\int (6x+7)\sin 2x\,dx$. Check your answer by differentiating.

ANSWER

$-\frac{1}{2}(6x+7)\cos 2x + \frac{3}{2}\sin 2x + C$

SOLUTION

Take $u = 6x+7$, which differentiation turns into a constant, and $dv = \sin 2x\,dx$. Then $du = 6\,dx$ and $v = -\frac{1}{2}\cos 2x$, so

Free sample from math.wiki for classroom use. Do not post. math.wiki

$$\int (6x+7)\sin 2x\,dx = -\frac{1}{2}(6x+7)\cos 2x + 3\int \cos 2x\,dx$$
$$= -\frac{1}{2}(6x+7)\cos 2x + \frac{3}{2}\sin 2x + C.$$

Check: the derivative is $-3\cos 2x + (6x+7)\sin 2x + 3\cos 2x = (6x+7)\sin 2x$.

3. Evaluate each definite integral exactly, using the definite form of integration by parts.

(a) $\displaystyle\int_0^{\pi/2} x\cos 2x\,dx$

(b) $\displaystyle\int_1^3 \ln(2x-1)\,dx$

ANSWER

(a) $-\dfrac{1}{2}$ (b) $\dfrac{5}{2}\ln 5 - 2$

SOLUTION

(a) Take $u = x$ and $dv = \cos 2x\,dx$, so $du = dx$ and $v = \dfrac{1}{2}\sin 2x$:

$$\int_0^{\pi/2} x\cos 2x\,dx = \Big[\frac{x}{2}\sin 2x\Big]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = 0 - \frac{1}{2}\Big[-\frac{1}{2}\cos 2x\Big]_0^{\pi/2} = -\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = -\frac{1}{2}.$$

The bracket is 0 at both ends since $\sin\pi = \sin 0 = 0$. The answer is negative because $\cos 2x < 0$ on $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$, where the factor x is largest.

(b) Take $u = \ln(2x-1)$ and $dv = dx$, so $du = \dfrac{2}{2x-1}\,dx$. Any antiderivative of dx serves as v; the choice $v = x - \dfrac{1}{2} = \dfrac{2x-1}{2}$ cancels the denominator, since $v\,du = dx$:

$$\int_1^3 \ln(2x-1)\,dx = \Big[\left(x-\tfrac{1}{2}\right)\ln(2x-1)\Big]_1^3 - \int_1^3 dx = \frac{5}{2}\ln 5 - \frac{1}{2}\ln 1 - 2 = \frac{5}{2}\ln 5 - 2.$$

(With $v = x$ instead, $\int_1^3 \dfrac{2x}{2x-1}\,dx = \int_1^3\left(1 + \dfrac{1}{2x-1}\right)dx = 2 + \dfrac{1}{2}\ln 5$, and $3\ln 5 - 2 - \dfrac{1}{2}\ln 5$ gives the same value.) The value is about 2.02.

4. Use tabular integration to find $\displaystyle\int x^2(x-3)^6\,dx$. Keep $(x-3)^6$ as the factor you integrate; do not expand it.

ANSWER

$$\frac{x^2(x-3)^7}{7} - \frac{x(x-3)^8}{28} + \frac{(x-3)^9}{252} + C$$

SOLUTION

The polynomial x^2 dies out after three derivatives, and $(x-3)^6$ integrates by the power rule: $\dfrac{(x-3)^7}{7}$, $\dfrac{(x-3)^8}{56}$, $\dfrac{(x-3)^9}{504}$.

sign	x^2 and its derivatives	$(x-3)^6$ and its antiderivatives
$+$	x^2	$(x-3)^6$
$-$	$2x$	$\dfrac{1}{7}(x-3)^7$

Free sample from math.wiki for classroom use. Do not post. math.wiki

sign	x^2 and its derivatives	$(x-3)^6$ and its antiderivatives
$+$	2	$\frac{1}{56}(x-3)^8$
	0	$\frac{1}{504}(x-3)^9$

$$\int x^2(x-3)^6\,dx = \frac{x^2(x-3)^7}{7} - \frac{2x(x-3)^8}{56} + \frac{2(x-3)^9}{504} + C = \frac{x^2(x-3)^7}{7} - \frac{x(x-3)^8}{28} + \frac{(x-3)^9}{252} + C.$$

Expanding $(x-3)^6$ and integrating eight terms would give the same function in another form.

5. Let $I = \int e^{-x}\sin 3x\,dx$.

(a) Integrate by parts twice, keeping $e^{-x}\,dx$ as dv both times, and solve the resulting equation for I.

(b) Use (a) to evaluate $\int_0^{\pi} e^{-x}\sin 3x\,dx$ exactly.

ANSWER

(a) $I = -\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x) + C$ (b) $\frac{3}{10}(1+e^{-\pi})$

SOLUTION

(a) Take $u = \sin 3x$ and $dv = e^{-x}\,dx$, so $du = 3\cos 3x\,dx$ and $v = -e^{-x}$:

$$I = -e^{-x}\sin 3x + 3\int e^{-x}\cos 3x\,dx.$$

Again with $u = \cos 3x$ and $dv = e^{-x}\,dx$, so $du = -3\sin 3x\,dx$:

$$\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x - 3I.$$

Substituting, $I = -e^{-x}\sin 3x - 3e^{-x}\cos 3x - 9I$, so $10I = -e^{-x}(\sin 3x + 3\cos 3x)$ and

$$I = -\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x) + C.$$

Check: the derivative is $\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x) - \frac{e^{-x}}{10}(3\cos 3x - 9\sin 3x) = e^{-x}\sin 3x$.

(b) With $F(x) = -\frac{e^{-x}}{10}(\sin 3x + 3\cos 3x)$: $F(\pi) = -\frac{e^{-\pi}}{10}(0-3) = \frac{3e^{-\pi}}{10}$ and $F(0) = -\frac{3}{10}$. So

$$\int_0^{\pi} e^{-x}\sin 3x\,dx = \frac{3e^{-\pi}}{10} + \frac{3}{10} = \frac{3}{10}\left(1+e^{-\pi}\right) \approx 0.313.$$

6. This problem finds $\int x\ln(x+1)\,dx$ for $x > -1$ in two ways, with $u = \ln(x+1)$ and $dv = x\,dx$ both times.

(a) Use $v = \frac{x^2}{2}$. The new integral needs long division.

(b) Use $v = \frac{x^2-1}{2}$ instead. Explain why this v is allowed, and finish.

(c) Show that your two answers agree.

ANSWER

Free sample from math.wiki for classroom use. Do not post. math.wiki

(a) $\dfrac{x^2}{2}\ln(x+1) - \dfrac{x^2}{4} + \dfrac{x}{2} - \dfrac{1}{2}\ln(x+1) + C$ (b) $\dfrac{x^2-1}{2}\ln(x+1) - \dfrac{x^2}{4} + \dfrac{x}{2} + C$ (c) the two logarithm terms of (a) combine into the one of (b)

SOLUTION

Here $du = \dfrac{1}{x+1}\,dx$.

(a) With $v = \dfrac{x^2}{2}$,

$$\int x\ln(x+1)\,dx = \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,dx.$$

Long division gives $\dfrac{x^2}{x+1} = x - 1 + \dfrac{1}{x+1}$, so the new integral is $\dfrac{x^2}{2} - x + \ln(x+1)$ (no bars, since $x+1>0$), and

$$\int x\ln(x+1)\,dx = \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1) + C.$$

(b) Any antiderivative of $dv = x\,dx$ may serve as v, and $\dfrac{x^2-1}{2} = \dfrac{x^2}{2} - \dfrac{1}{2}$ is one, since its derivative is x. It is worth choosing because $x^2-1 = (x-1)(x+1)$ cancels the denominator of du:

$$\int x\ln(x+1)\,dx = \frac{x^2-1}{2}\ln(x+1) - \int \frac{(x-1)(x+1)}{2(x+1)}\,dx = \frac{x^2-1}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} + C.$$

(c) In (a), $\dfrac{x^2}{2}\ln(x+1) - \dfrac{1}{2}\ln(x+1) = \dfrac{x^2-1}{2}\ln(x+1)$, so the two answers are the same family of functions. Check: the derivative of (b) is $x\ln(x+1) + \dfrac{x^2-1}{2(x+1)} - \dfrac{x}{2} + \dfrac{1}{2} = x\ln(x+1) + \dfrac{x-1}{2} - \dfrac{x-1}{2} = x\ln(x+1)$.

7. Arjun's astronomy club counts meteors during a shower. In a made-up model, meteors arrive at the rate $r(t) = 30t^2e^{-t}$ meteors per hour, where t is the number of hours after midnight.

(a) Find a formula for $N(T)$, the number of meteors counted from midnight to T hours after midnight.

(b) Find the exact number counted in the first 4 hours. Then use a calculator and round to the nearest whole meteor.

(c) Leah joins the count at 1 a.m. and leaves at 3 a.m. Find the exact number of meteors in her window, then use a calculator and round to the nearest whole meteor.

ANSWER

(a) $N(T) = 60 - 30e^{-T}\left(T^2 + 2T + 2\right)$ (b) $60 - 780e^{-4}$, about 46 meteors (c) $150e^{-1} - 510e^{-3}$, about 30 meteors

SOLUTION

(a) An antiderivative of $30t^2e^{-t}$ by a table: differentiate $30t^2$ to $60t$, 60, 0; integrate e^{-t} to $-e^{-t}$, e^{-t}, $-e^{-t}$. With signs $+, -, +$:

$$\int 30t^2e^{-t}\,dt = -30t^2e^{-t} - 60te^{-t} - 60e^{-t} + C = -30e^{-t}\left(t^2 + 2t + 2\right) + C.$$

Free sample from math.wiki for classroom use. Do not post. math.wiki

So

$$N(T) = \int_0^T 30t^2 e^{-t}\,dt = -30e^{-T}\left(T^2 + 2T + 2\right) + 30 \cdot 2 = 60 - 30e^{-T}\left(T^2 + 2T + 2\right).$$

Check: $N(0) = 60 - 60 = 0$.
(b) $N(4) = 60 - 30e^{-4}(16 + 8 + 2) = 60 - 780e^{-4} \approx 60 - 14.29 = 45.71$, so about 46 meteors.
(c) Her count is $N(3) - N(1) = 30e^{-1}(1 + 2 + 2) - 30e^{-3}(9 + 6 + 2) = 150e^{-1} - 510e^{-3} \approx 55.18 - 25.39 = 29.79$, so about 30 meteors. Her two hours around the peak at $t = 2$ hold about two thirds of the first four hours' count.

Free sample from math.wiki for classroom use. Do not post. math.wiki