Calculus 2 Unit 1 worksheet: Integration by partial fractions (answer key)

ANSWERS AT A GLANCE

1. (a) $3\ln|x-2| - \ln|x+1| + C$ (b) $\ln|x-2| - \ln|3x+1| + C$ (c) $\frac{x^2}{2} - x + \frac{1}{2}\ln|x-1| + \frac{9}{2}\ln|x+3| + C$
2. $-3\ln 3 + 5\ln 2$
3. (a) $2\ln 5 - 5\ln 2 + \frac{3}{2}$ (b) $\ln|x-1| - \frac{2}{x-1} - \frac{1}{(x-1)^2} + C$
4. $-\frac{3}{2}\ln 2 - \frac{\pi}{4}$ (about -1.825); negative because the top and the quadratic factor are positive and $x-1$ is negative on $[-1, 0]$.
5. $\frac{1}{2}\ln\left(x^2-2x+5\right) + \arctan\frac{x-1}{2} - \frac{3}{x^2-2x+5} + C$
6. (a) $f(x) = \frac{2}{x+1} + \frac{3}{(x-2)^2} + \frac{1}{x^2+1}$ (b) $(x+1)(x-2)^2\left(x^2+1\right)$, degree 5, so 5 unknown constants; cover-up finds the constant over $x+1$ and the constant over $(x-2)^2$. (c) $(-\infty, -1)$, $(-1, 2)$ and $(2, \infty)$
7. (a) $24 - 6\ln 3 - 12\ln 2$ hundred cartons, about 909 cartons. (b) $r(t) = 12 - \frac{6}{t+1} - \frac{12}{t+2}$, and both fractions are positive for $t > 0$, so $r(t) < 12$ hundred cartons per hour.

WORKED SOLUTIONS

1. Find each integral by partial fractions. Divide first where the fraction is not proper.

(a) $\displaystyle\int \frac{2x+5}{x^2-x-2}\,dx$ (b) $\displaystyle\int \frac{7}{3x^2-5x-2}\,dx$

(c) $\displaystyle\int \frac{x^3+x^2}{x^2+2x-3}\,dx$

ANSWER

(a) $3\ln|x-2| - \ln|x+1| + C$ (b) $\ln|x-2| - \ln|3x+1| + C$ (c) $\frac{x^2}{2} - x + \frac{1}{2}\ln|x-1| + \frac{9}{2}\ln|x+3| + C$

SOLUTION

(a) $x^2 - x - 2 = (x-2)(x+1)$ and the top has degree $1 < 2$. Cover-up: at $x = 2$, covering $x-2$ leaves $\frac{2x+5}{x+1} = \frac{9}{3} = 3$; at $x = -1$, covering $x+1$ leaves $\frac{2x+5}{x-2} = \frac{3}{-3} = -1$. Check: $3(x+1) - (x-2) = 2x+5$. So the integral is $3\ln|x-2| - \ln|x+1| + C$.

(b) $3x^2 - 5x - 2 = (3x+1)(x-2)$. Cover-up: at $x = -\frac{1}{3}$, covering $3x+1$ leaves $\frac{7}{x-2} = \frac{7}{-7/3} = -3$; at $x = 2$, covering $x-2$ leaves $\frac{7}{3x+1} = 1$. So $\frac{7}{3x^2-5x-2} = \frac{-3}{3x+1} + \frac{1}{x-2}$. With $a = 3$, $\int \frac{-3}{3x+1}\,dx = -\ln|3x+1| + C$, so the integral is $\ln|x-2| - \ln|3x+1| + C$.

(c) The top has degree 3, so divide: $x^3 + x^2 = (x-1)\left(x^2+2x-3\right) + 5x - 3$, since $(x-1)\left(x^2+2x-3\right) = x^3 + x^2 - 5x + 3$. The remainder is proper and $x^2 + 2x - 3 = (x+3)(x-1)$. Cover-up on $\frac{5x-3}{(x+3)(x-1)}$: at $x = 1$, $\frac{2}{4} = \frac{1}{2}$; at $x = -3$, $\frac{-18}{-4} = \frac{9}{2}$. So

$$\int \frac{x^3+x^2}{x^2+2x-3}\,dx = \frac{x^2}{2} - x + \frac{1}{2}\ln|x-1| + \frac{9}{2}\ln|x+3| + C.$$

Each answer holds on each interval between zeros of the bottom.

2. Evaluate $\displaystyle\int_0^2 \frac{2x-1}{x^2+3x+2}\,dx$. Give the exact value.

ANSWER

$-3\ln 3 + 5\ln 2$

SOLUTION

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The bottom factors as $(x+1)(x+2)$, two distinct linear factors, and the top has degree $1 < 2$. Cover-up: at $x=-1$, covering $x+1$ leaves $\dfrac{2x-1}{x+2}=\dfrac{-3}{1}=-3$; at $x=-2$, covering $x+2$ leaves $\dfrac{2x-1}{x+1}=\dfrac{-5}{-1}=5$. So the integrand is $\dfrac{-3}{x+1}+\dfrac{5}{x+2}$.

Both factors are positive on $[0,2]$, so the integrand is continuous there and

$$\int_0^2 \frac{2x-1}{x^2+3x+2}\,dx = \Big[-3\ln(x+1)+5\ln(x+2)\Big]_0^2 = -3\ln 3+5\ln 2,$$

since at the upper limit the logs are $\ln 3$ and $\ln 4$, and at 0 they are $\ln 1$ and $\ln 2$.

3. Each bottom has a repeated linear factor.

(a) Evaluate $\displaystyle\int_3^4 \frac{x^2-4x+13}{x^3-3x^2+4}\,dx$ exactly. (Look for a zero of the bottom among small whole numbers.)

(b) Find $\displaystyle\int \frac{x^2+1}{(x-1)^3}\,dx$.

ANSWER

(a) $2\ln 5-5\ln 2+\dfrac{3}{2}$ (b) $\ln|x-1|-\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}+C$

SOLUTION

(a) At $x=-1$ the bottom is $-1-3+4=0$, so $x+1$ is a factor, and $x^3-3x^2+4=(x+1)\left(x^2-4x+4\right)=(x+1)(x-2)^2$. So

$$\frac{x^2-4x+13}{(x+1)(x-2)^2}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}.$$

Cover-up: at $x=-1$, $\dfrac{1+4+13}{9}=2=A$; at $x=2$, covering $(x-2)^2$ leaves $\dfrac{4-8+13}{3}=3=C$. Matching x^2 in $x^2-4x+13=A(x-2)^2+B(x+1)(x-2)+C(x+1)$ gives $1=A+B$, so $B=-1$. An antiderivative is $F(x)=2\ln|x+1|-\ln|x-2|-\dfrac{3}{x-2}$, and $[3,4]$ contains neither -1 nor 2, so

$$F(4)-F(3)=\left(2\ln 5-\ln 2-\frac{3}{2}\right)-(2\ln 4-\ln 1-3)=2\ln 5-5\ln 2+\frac{3}{2}\approx 1.253.$$

(b) The bottom is one linear factor to the third power, so the form is $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}$. A quick way to the constants: write the top in powers of $u=x-1$. With $x=u+1$, $x^2+1=u^2+2u+2$, so

$$\frac{x^2+1}{(x-1)^3}=\frac{u^2+2u+2}{u^3}=\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{2}{(x-1)^3}.$$

By the power rule, $\displaystyle\int\frac{2}{(x-1)^2}\,dx=-\frac{2}{x-1}$ and $\displaystyle\int\frac{2}{(x-1)^3}\,dx=-\frac{1}{(x-1)^2}$, so the integral is $\ln|x-1|-\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}+C$, on $x<1$ and on $x>1$.

4. The bottom of this integrand has a zero at a small whole number. Evaluate the integral exactly, and explain why the sign of your answer makes sense.

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$$\int_{-1}^{0} \frac{3x^2+3x+4}{x^3+x^2-2}\,dx$$

ANSWER
$-\frac{3}{2}\ln 2 - \frac{\pi}{4}$ (about −1.825); negative because the top and the quadratic factor are positive and $x-1$ is negative on $[-1,0]$.

SOLUTION
At $x=1$ the bottom is 1 + 1 − 2 = 0, so $x-1$ is a factor, and $x^3+x^2-2=(x-1)\left(x^2+2x+2\right)$. The quadratic has discriminant 4 − 8 < 0, so it is irreducible:

$$\frac{3x^2+3x+4}{(x-1)\left(x^2+2x+2\right)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+2x+2}.$$

Cover-up at $x=1$: $A=\frac{10}{5}=2$. Clearing denominators, $3x^2+3x+4=2\left(x^2+2x+2\right)+(Bx+C)(x-1)$. Matching x^2: $3=2+B$, so $B=1$. Matching the constant: $4=4-C$, so $C=0$. So the second piece is $\frac{x}{x^2+2x+2}$. Split the top as a multiple of $2x+2$ plus a constant: $x=\frac{1}{2}(2x+2)-1$, and $x^2+2x+2=(x+1)^2+1$. An antiderivative is

$$F(x)=2\ln|x-1|+\frac{1}{2}\ln\left(x^2+2x+2\right)-\arctan(x+1).$$

The only real zero of the bottom is 1, outside $[-1,0]$, so

$$F(0)-F(-1)=\left(0+\frac{1}{2}\ln 2-\frac{\pi}{4}\right)-(2\ln 2+0-0)=-\frac{3}{2}\ln 2-\frac{\pi}{4}\approx -1.825.$$

The sign makes sense: the top $3x^2+3x+4$ has negative discriminant and is always positive, $x^2+2x+2>0$, and $x-1<0$ on $[-1,0]$, so the integrand is negative on the whole interval.

5. The bottom of this integrand is a repeated irreducible quadratic. Find the integral.

$$\int \frac{x^3-x^2+9x-1}{\left(x^2-2x+5\right)^2}\,dx$$

ANSWER
$\frac{1}{2}\ln\left(x^2-2x+5\right)+\arctan\frac{x-1}{2}-\frac{3}{x^2-2x+5}+C$

SOLUTION
x^2-2x+5 has discriminant 4 − 20 < 0, so it is irreducible, and it appears twice:

$$\frac{x^3-x^2+9x-1}{\left(x^2-2x+5\right)^2} = \frac{Ax+B}{x^2-2x+5}+\frac{Cx+D}{\left(x^2-2x+5\right)^2}.$$

Clearing denominators, $x^3-x^2+9x-1=(Ax+B)\left(x^2-2x+5\right)+Cx+D=Ax^3+(B-2A)x^2+(5A-2B+C)x+(5B+D)$. Matching: $A=1$; $B-2=-1$, so $B=1$; $5-2+C=9$, so $C=6$; $5+D=-1$, so $D=-6$.

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First piece: $x+1=\dfrac{1}{2}(2x-2)+2$ and $x^2-2x+5=(x-1)^2+4$, so

$$\int \frac{x+1}{x^2-2x+5}\,dx=\frac{1}{2}\ln\left(x^2-2x+5\right)+2\cdot\frac{1}{2}\arctan\frac{x-1}{2}.$$

Second piece: $6x-6=3(2x-2)$ is a multiple of the derivative of the quadratic, so with $u=x^2-2x+5$ it is $\int 3u^{-2}\,du=-\dfrac{3}{u}$. No constant is left over, so no trigonometric substitution is needed. Altogether

$$\int \frac{x^3-x^2+9x-1}{\left(x^2-2x+5\right)^2}\,dx=\frac{1}{2}\ln\left(x^2-2x+5\right)+\arctan\frac{x-1}{2}-\frac{3}{x^2-2x+5}+C,$$

for every real x (no absolute value is needed, since the quadratic is always positive).

6. A rational function f has the antiderivative

$$F(x)=2\ln|x+1|-\frac{3}{x-2}+\arctan x.$$

(a) Write f as a sum of partial fractions.

(b) Write the bottom of f, as one fraction in lowest terms, in factored form. If you were given f as one fraction, how many unknown constants would its decomposition have, and which of them could cover-up find directly?

(c) On which intervals is F an antiderivative of f?

ANSWER

(a) $f(x)=\dfrac{2}{x+1}+\dfrac{3}{(x-2)^2}+\dfrac{1}{x^2+1}$ (b) $(x+1)(x-2)^2\left(x^2+1\right)$, degree 5, so 5 unknown constants; cover-up finds the constant over $x+1$ and the constant over $(x-2)^2$. (c) $(-\infty,-1)$, $(-1,2)$ and $(2,\infty)$

SOLUTION

(a) Differentiate each term: $\dfrac{d}{dx}\,2\ln|x+1|=\dfrac{2}{x+1}$, $\dfrac{d}{dx}\left(-3(x-2)^{-1}\right)=3(x-2)^{-2}$, and $\dfrac{d}{dx}\arctan x=\dfrac{1}{x^2+1}$. So $f(x)=\dfrac{2}{x+1}+\dfrac{3}{(x-2)^2}+\dfrac{1}{x^2+1}$.

(b) The common bottom is $(x+1)(x-2)^2\left(x^2+1\right)$, which has degree 1 + 2 + 2 = 5. (The top is $2x^4-4x^3+10x^2-5x+15$, which is not 0 at -1 or at 2, so nothing cancels.) The full form has one constant over $x+1$, two over the powers of $x-2$ (one of them 0 here), and $Bx+C$ over x^2+1: 1 + 2 + 2 = 5 constants, the degree of the bottom. Cover-up finds the constant over $x+1$ (put $x=-1$) and the one over the highest power $(x-2)^2$ (put $x=2$). The constant over $x-2$ and the two over x^2+1 need matching coefficients.

(c) F breaks where $\ln|x+1|$ or $\dfrac{3}{x-2}$ does, at $x=-1$ and $x=2$; $\arctan x$ is defined everywhere. So F is an antiderivative of f on $(-\infty,-1)$, $(-1,2)$ and $(2,\infty)$, and the constant C may differ from one interval to the next.

7. Yuna runs a carton-folding machine at a packaging plant. After a restart, in a made-up model, it folds cartons at the rate

$$r(t)=\frac{12t^2+18t}{t^2+3t+2}$$

hundred cartons per hour, t hours after the restart.

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(a) Find the exact number of hundreds of cartons folded in the first 2 hours. Then use a calculator to give the number of cartons, rounded to the nearest whole carton.

(b) Use your work in (a) to explain why the rate stays below 1,200 cartons per hour for all $t > 0$.

ANSWER

(a) $24 - 6\ln 3 - 12\ln 2$ hundred cartons, about 909 cartons. (b) $r(t) = 12 - \dfrac{6}{t+1} - \dfrac{12}{t+2}$, and both fractions are positive for $t > 0$, so $r(t) < 12$ hundred cartons per hour.

SOLUTION

(a) The top and bottom both have degree 2, so divide: $12t^2 + 18t = 12\left(t^2 + 3t + 2\right) - 18t - 24$, and

$$r(t) = 12 - \frac{18t + 24}{(t+1)(t+2)}.$$

Cover-up on $\dfrac{18t+24}{(t+1)(t+2)}$: at $t = -1$, $\dfrac{6}{1} = 6$; at $t = -2$, $\dfrac{-12}{-1} = 12$. So $r(t) = 12 - \dfrac{6}{t+1} - \dfrac{12}{t+2}$. The folded amount is

$$\int_0^2 r(t)\,dt = \Big[12t - 6\ln(t+1) - 12\ln(t+2)\Big]_0^2 = 24 - 6\ln 3 - 12\ln\frac{4}{2} = 24 - 6\ln 3 - 12\ln 2$$

hundred cartons, that is $2{,}400 - 600\ln 3 - 1{,}200\ln 2 \approx 909.06$, about 909 cartons.

(b) For $t > 0$, $\dfrac{6}{t+1} > 0$ and $\dfrac{12}{t+2} > 0$, so $r(t) = 12 - \dfrac{6}{t+1} - \dfrac{12}{t+2} < 12$ hundred, that is below 1,200 cartons per hour. (Both fractions shrink as t grows, so the rate climbs toward 12 hundred without reaching it.)

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