Calculus 2 Unit 1 worksheet: Trigonometric integrals (answer key)

ANSWERS AT A GLANCE

1. (a) $-\frac{\cos 2x}{2}+\frac{\cos^3 2x}{6}+C$ (b) $\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+C$ (c) $-\csc x-\sin x+C$ (d) $\frac{\sec^3 x}{3}-\sec x+C$
2. (a) $\frac{2}{3}\tan^{3/2} x+\frac{2}{7}\tan^{7/2} x+C$ (b) $\frac{\sec^7 x}{7}-\frac{2\sec^5 x}{5}+\frac{\sec^3 x}{3}+C$ (c) $\frac{\tan^5 x}{5}-\frac{\tan^3 x}{3}+\tan x-x+C$
3. $-\frac{\cos 8x}{16}-\frac{\cos 2x}{4}+C$
4. $\frac{3\pi}{32}+\frac{1}{4}$, about 0.545
5. $\frac{7}{24}$
6. (a) $\frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$ (b) $\frac{\sin^4 x}{4}-\frac{\sin^6 x}{6}+C$ (c) the difference is $-\frac{1}{12}$ for every x.

WORKED SOLUTIONS

1. Find each integral. Say which factor you set aside for du.

(a) $\int \sin^3 2x\,dx$

(b) $\int \sin^5 x\cos^3 x\,dx$

(c) $\int \frac{\cos^3 x}{\sin^2 x}\,dx$

(d) $\int \frac{\sin^3 x}{\cos^4 x}\,dx$

ANSWER

(a) $-\frac{\cos 2x}{2}+\frac{\cos^3 2x}{6}+C$ (b) $\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+C$ (c) $-\csc x-\sin x+C$ (d) $\frac{\sec^3 x}{3}-\sec x+C$

SOLUTION

(a) The power of sine is odd and the angle is $2x$. Set one $\sin 2x$ aside: $\sin^3 2x=\left(1-\cos^2 2x\right)\sin 2x$. With $u=\cos 2x$, $du=-2\sin 2x\,dx$, so $\sin 2x\,dx=-\frac{1}{2}\,du$:

$$-\frac{1}{2}\int\left(1-u^2\right)du=-\frac{u}{2}+\frac{u^3}{6}+C=-\frac{\cos 2x}{2}+\frac{\cos^3 2x}{6}+C.$$

(b) Both powers are odd, so either route works; setting one $\cos x$ aside leaves the smaller power to convert. $\sin^5 x\cos^3 x=\sin^5 x\left(1-\sin^2 x\right)\cos x$, and with $u=\sin x$, $du=\cos x\,dx$:

$$\int\left(u^5-u^7\right)du=\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+C.$$

(Setting $\sin x$ aside instead gives an answer that differs from this one by a constant.)

(c) The power of cosine is odd; the power of sine is -2, which plays no part in the rewrite. $\frac{\cos^3 x}{\sin^2 x}=\left(1-\sin^2 x\right)\sin^{-2} x\,\cos x$. With $u=\sin x$:

$$\int\left(u^{-2}-1\right)du=-\frac{1}{u}-u+C=-\csc x-\sin x+C,$$

on each interval where $\sin x\neq 0$.

(d) The power of sine is odd. $\frac{\sin^3 x}{\cos^4 x}=\left(1-\cos^2 x\right)\cos^{-4} x\,\sin x$. With $u=\cos x$, $\sin x\,dx=-du$:

$$-\int\left(u^{-4}-u^{-2}\right)du=\frac{u^{-3}}{3}-u^{-1}+C=\frac{\sec^3 x}{3}-\sec x+C,$$

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on each interval where $\cos x \neq 0$. Check: the derivative is $\sec^3 x \tan x - \sec x \tan x = \sec x \tan x \left(\sec^2 x - 1\right) = \sec x \tan^3 x = \dfrac{\sin^3 x}{\cos^4 x}$.

2. Find each integral. Say which factor you set aside for du, or how you peeled off a power.

(a) $\int \sqrt{\tan x}\, \sec^4 x\, dx$ (b) $\int \tan^5 x \sec^3 x\, dx$

(c) $\int \tan^6 x\, dx$

ANSWER

(a) $\dfrac{2}{3}\tan^{3/2} x + \dfrac{2}{7}\tan^{7/2} x + C$ (b) $\dfrac{\sec^7 x}{7} - \dfrac{2\sec^5 x}{5} + \dfrac{\sec^3 x}{3} + C$ (c) $\dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x + C$

SOLUTION

(a) The power of secant is even, and the power of tangent, $\dfrac{1}{2}$, may be anything. Set $\sec^2 x$ aside and convert $\sec^2 x = 1 + \tan^2 x$: $\tan^{1/2} x \left(1 + \tan^2 x\right) \sec^2 x$. With $u = \tan x$, $du = \sec^2 x\, dx$:

$$\int \left(u^{1/2} + u^{5/2}\right) du = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C,$$

on each interval where $\tan x > 0$.

(b) The power of secant is odd, so the even-secant route is closed; the power of tangent is odd. Set $\sec x \tan x$ aside: $\tan^4 x \sec^2 x \left(\sec x \tan x\right) = \left(\sec^2 x - 1\right)^2 \sec^2 x \left(\sec x \tan x\right)$. With $u = \sec x$, $du = \sec x \tan x\, dx$:

$$\int \left(u^2 - 1\right)^2 u^2\, du = \int \left(u^6 - 2u^4 + u^2\right) du = \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C.$$

(c) Tangent alone: peel off $\tan^2 x = \sec^2 x - 1$ each time.

$$\int \tan^6 x\, dx = \frac{\tan^5 x}{5} - \int \tan^4 x\, dx, \qquad \int \tan^4 x\, dx = \frac{\tan^3 x}{3} - \int \tan^2 x\, dx,$$

and $\int \tan^2 x\, dx = \int \left(\sec^2 x - 1\right) dx = \tan x - x$. Since the power is even, the chain ends at $\int dx$, not at $\int \tan x\, dx$:

$$\int \tan^6 x\, dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C.$$

3. Find $\displaystyle\int \sin 5x \cos 3x\, dx$.

ANSWER

$-\dfrac{\cos 8x}{16} - \dfrac{\cos 2x}{4} + C$

SOLUTION

The angles differ, so use the product to sum formula $\sin A \cos B = \dfrac{1}{2}\big[\sin(A+B) + \sin(A-B)\big]$ with $A = 5x$ and $B = 3x$:

$$\sin 5x \cos 3x = \tfrac{1}{2}\sin 8x + \tfrac{1}{2}\sin 2x.$$

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With $\int \sin kx\,dx = -\dfrac{\cos kx}{k} + C$ for each term:

$$\int \sin 5x \cos 3x\,dx = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C.$$

4. Evaluate $\int_0^{\pi/4} \cos^4 x\,dx$ exactly.

ANSWER

$\dfrac{3\pi}{32} + \dfrac{1}{4}$, about 0.545

SOLUTION

The power is even, so lower it. With $\cos^2 x = \dfrac{1 + \cos 2x}{2}$,

$$\cos^4 x = \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right).$$

The last term is an even power again: $\cos^2 2x = \dfrac{1 + \cos 4x}{2}$. So

$$\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$$

Integrate term by term:

$$\int_0^{\pi/4} \cos^4 x\,dx = \left[\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right]_0^{\pi/4} = \frac{3\pi}{32} + \frac{1}{4} + 0.$$

Unlike a whole period, the interval $\left[0, \dfrac{\pi}{4}\right]$ does not cancel the $\cos 2x$ term: $\sin\dfrac{\pi}{2} = 1$ leaves $\dfrac{1}{4}$. The value is about 0.545, less than the interval's length $\dfrac{\pi}{4} \approx 0.785$, as it must be since $0 \le \cos^4 x \le 1$.

5. Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan^5 x \sec^4 x\,dx$ exactly.

ANSWER

$\dfrac{7}{24}$

SOLUTION

The power of secant is even. Set $\sec^2 x$ aside and convert the other $\sec^2 x$ to $1 + \tan^2 x$. With $u = \tan x$, $du = \sec^2 x\,dx$, and the limits change: $\tan 0 = 0$ and $\tan\dfrac{\pi}{4} = 1$. So

$$\int_0^1 u^5\left(1 + u^2\right) du = \left[\frac{u^6}{6} + \frac{u^8}{8}\right]_0^1 = \frac{7}{24}.$$

The value is positive, as it must be: $\tan x$ and $\sec x$ are positive on $\left(0, \dfrac{\pi}{4}\right)$.

6. Naomi and Diego each find $\int \sin^3 x \cos^3 x\,dx$. Naomi sets one factor $\sin x$ aside; Diego sets one factor $\cos x$ aside.

(a) Carry out Naomi's route.

(b) Carry out Diego's route.

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(c) The two antiderivatives look different. Show that Naomi's minus Diego's is a constant, and find that constant.

ANSWER

(a) $\dfrac{\cos^6 x}{6} - \dfrac{\cos^4 x}{4} + C$ (b) $\dfrac{\sin^4 x}{4} - \dfrac{\sin^6 x}{6} + C$ (c) the difference is $-\dfrac{1}{12}$ for every x.

SOLUTION

(a) $\sin^3 x \cos^3 x = \left(1 - \cos^2 x\right) \cos^3 x \, \sin x$. With $u = \cos x$, $\sin x \, dx = -du$:

$$-\int \left(u^3 - u^5\right) du = -\frac{u^4}{4} + \frac{u^6}{6} + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C.$$

(b) $\sin^3 x \cos^3 x = \sin^3 x \left(1 - \sin^2 x\right) \cos x$. With $u = \sin x$, $du = \cos x \, dx$:

$$\int \left(u^3 - u^5\right) du = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C.$$

(c) Write $c = \cos^2 x$, so $\sin^2 x = 1 - c$. Then

$$\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} = \frac{1 - 2c + c^2}{4} - \frac{1 - 3c + 3c^2 - c^3}{6} = \frac{1}{12} - \frac{c^2}{4} + \frac{c^3}{6},$$

since the terms in c cancel ($-\dfrac{c}{2} + \dfrac{c}{2}$). Naomi's antiderivative is $\dfrac{c^3}{6} - \dfrac{c^2}{4}$. So Naomi's minus Diego's is

$$\left(\frac{c^3}{6} - \frac{c^2}{4}\right) - \left(\frac{1}{12} - \frac{c^2}{4} + \frac{c^3}{6}\right) = -\frac{1}{12},$$

the same for every x. Both are correct; the constant C absorbs the difference. A quick test: at $x = 0$, Naomi's gives $\dfrac{1}{6} - \dfrac{1}{4} = -\dfrac{1}{12}$ and Diego's gives 0.

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