Calculus 2 Unit 1 worksheet: Trigonometric substitution (answer key)

ANSWERS AT A GLANCE

1. $-\dfrac{\left(16-x^2\right)^{3/2}}{48x^3}+C$, on $-4<x<0$ and on $0<x<4$ separately.
2. $\ln\left(x+\sqrt{x^2+16}\right)-\dfrac{x}{\sqrt{x^2+16}}+C$
3. $-\dfrac{x}{16\sqrt{x^2-16}}+C$
4. $\dfrac{x-4}{16\sqrt{8x-x^2}}+C$
5. $\sqrt{x^2+8x+20}-4\ln\left(x+4+\sqrt{x^2+8x+20}\right)+C$
6. $\dfrac{11}{6144}$
7. The value is $\ln\left(2+\sqrt{3}\right)-\dfrac{\sqrt{3}}{2}$ for every $a>0$ (about 0.451).

WORKED SOLUTIONS

1. Find $\displaystyle\int \frac{\sqrt{16-x^2}}{x^4}\,dx$.

ANSWER

$-\dfrac{\left(16-x^2\right)^{3/2}}{48x^3}+C$, on $-4<x<0$ and on $0<x<4$ separately.

SOLUTION

The root has the shape $\sqrt{a^2-x^2}$ with $a=4$. Let $x=4\sin\theta$ with $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$ and $\theta\neq 0$ (since $x\neq 0$). Then $dx=4\cos\theta\,d\theta$ and $\sqrt{16-16\sin^2\theta}=4\cos\theta$, because $\cos\theta>0$ on this interval:

$$\int \frac{\sqrt{16-x^2}}{x^4}\,dx=\int \frac{4\cos\theta\cdot 4\cos\theta}{256\sin^4\theta}\,d\theta=\frac{1}{16}\int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\sin^2\theta}\,d\theta=\frac{1}{16}\int \cot^2\theta\csc^2\theta\,d\theta.$$

With $u=\cot\theta$, $du=-\csc^2\theta\,d\theta$, this is $-\dfrac{1}{16}\int u^2\,du=-\dfrac{\cot^3\theta}{48}+C$.

The triangle has hypotenuse 4, opposite side x and adjacent side $\sqrt{16-x^2}$, so $\cot\theta=\dfrac{\sqrt{16-x^2}}{x}$ (true for negative θ too, since $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$ and $\cos\theta=\dfrac{\sqrt{16-x^2}}{4}$). Therefore

$$\int \frac{\sqrt{16-x^2}}{x^4}\,dx=-\frac{\left(16-x^2\right)^{3/2}}{48x^3}+C,$$

on each side of $x=0$, with constants that need not match.

2. Find $\displaystyle\int \frac{x^2}{\left(x^2+16\right)^{3/2}}\,dx$.

ANSWER

$\ln\left(x+\sqrt{x^2+16}\right)-\dfrac{x}{\sqrt{x^2+16}}+C$

SOLUTION

The shape is x^2+a^2 with $a=4$. Let $x=4\tan\theta$ with $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$. Then $dx=4\sec^2\theta\,d\theta$ and $x^2+16=16\sec^2\theta$, whose $\dfrac{3}{2}$ power is $64\sec^3\theta$ because $\sec\theta>0$:

$$\int \frac{16\tan^2\theta\cdot 4\sec^2\theta}{64\sec^3\theta}\,d\theta=\int \frac{\tan^2\theta}{\sec\theta}\,d\theta=\int \frac{\sec^2\theta-1}{\sec\theta}\,d\theta=\int(\sec\theta-\cos\theta)\,d\theta.$$

This is $\ln|\sec\theta+\tan\theta|-\sin\theta+C$.

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The triangle has opposite side x, adjacent side 4 and hypotenuse $\sqrt{x^2+16}$, so $\sec\theta+\tan\theta=\dfrac{\sqrt{x^2+16}+x}{4}$ and $\sin\theta=\dfrac{x}{\sqrt{x^2+16}}$. The $-\ln 4$ joins the constant, and $x+\sqrt{x^2+16}>0$ for every x, so the bars can go:

$$\int\frac{x^2}{(x^2+16)^{3/2}}\,dx=\ln\left(x+\sqrt{x^2+16}\right)-\frac{x}{\sqrt{x^2+16}}+C.$$

3. Find $\displaystyle\int\frac{dx}{(x^2-16)^{3/2}}$ for $x>4$.

ANSWER

$$-\frac{x}{16\sqrt{x^2-16}}+C$$

SOLUTION

The shape is x^2-a^2 with $a=4$, on the piece $x>4$. Let $x=4\sec\theta$ with $0<\theta<\dfrac{\pi}{2}$. Then $dx=4\sec\theta\tan\theta\,d\theta$ and $x^2-16=16\tan^2\theta$, whose $\dfrac{3}{2}$ power is $64\tan^3\theta$ because $\tan\theta>0$ on this piece:

$$\int\frac{4\sec\theta\tan\theta}{64\tan^3\theta}\,d\theta=\frac{1}{16}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta=\frac{1}{16}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

With $u=\sin\theta$, $du=\cos\theta\,d\theta$, this is $\dfrac{1}{16}\int u^{-2}\,du=-\dfrac{1}{16\sin\theta}+C$.

The triangle has hypotenuse x, adjacent side 4 and opposite side $\sqrt{x^2-16}$, so $\sin\theta=\dfrac{\sqrt{x^2-16}}{x}$ and

$$\int\frac{dx}{(x^2-16)^{3/2}}=-\frac{x}{16\sqrt{x^2-16}}+C.$$

4. Find $\displaystyle\int\frac{dx}{(8x-x^2)^{3/2}}$ on $0<x<8$. Complete the square first.

ANSWER

$$\frac{x-4}{16\sqrt{8x-x^2}}+C$$

SOLUTION

Complete the square: $8x-x^2=16-\left(x^2-8x+16\right)=16-(x-4)^2$. This is a^2-u^2 with $a=4$ and $u=x-4$, so let $x-4=4\sin\theta$ with $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$. Then $dx=4\cos\theta\,d\theta$ and $16-16\sin^2\theta=16\cos^2\theta$, whose $\dfrac{3}{2}$ power is $64\cos^3\theta$ because $\cos\theta>0$:

$$\int\frac{4\cos\theta}{64\cos^3\theta}\,d\theta=\frac{1}{16}\int\sec^2\theta\,d\theta=\frac{1}{16}\tan\theta+C.$$

The triangle has hypotenuse 4, opposite side $x-4$ and adjacent side $\sqrt{16-(x-4)^2}=\sqrt{8x-x^2}$, so $\tan\theta=\dfrac{x-4}{\sqrt{8x-x^2}}$ and

$$\int\frac{dx}{(8x-x^2)^{3/2}}=\frac{x-4}{16\sqrt{8x-x^2}}+C.$$

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5. Find $\displaystyle\int \frac{x}{\sqrt{x^2+8x+20}}\,dx$. Complete the square first.

ANSWER

$\sqrt{x^2+8x+20}-4\ln\left(x+4+\sqrt{x^2+8x+20}\right)+C$

SOLUTION

Complete the square: $x^2+8x+20=(x+4)^2+4$. Let $u=x+4$, so $x=u-4$ and $dx=du$:

$$\int \frac{x}{\sqrt{x^2+8x+20}}\,dx=\int \frac{u}{\sqrt{u^2+4}}\,du-4\int \frac{du}{\sqrt{u^2+4}}.$$

In the first integral u is half the derivative of u^2+4, so it equals $\sqrt{u^2+4}$ plus a constant. In the second, let $u=2\tan\theta$ with $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, so $du=2\sec^2\theta\,d\theta$ and $\sqrt{u^2+4}=2\sec\theta$:

$$\int \frac{du}{\sqrt{u^2+4}}=\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1=\ln\frac{u+\sqrt{u^2+4}}{2}+C_1.$$

The $-\ln 2$ joins the constant, and $u+\sqrt{u^2+4}>0$. Returning to x:

$$\int \frac{x}{\sqrt{x^2+8x+20}}\,dx=\sqrt{x^2+8x+20}-4\ln\left(x+4+\sqrt{x^2+8x+20}\right)+C.$$

6. Evaluate $\displaystyle\int_0^{\frac{4\sqrt{3}}{3}} \frac{dx}{\left(x^2+16\right)^{5/2}}$ with a trigonometric substitution, changing the limits to θ.

ANSWER

$\dfrac{11}{6144}$

SOLUTION

Let $x=4\tan\theta$, so $dx=4\sec^2\theta\,d\theta$ and $x^2+16=16\sec^2\theta$, whose $\dfrac{5}{2}$ power is $1{,}024\sec^5\theta$ because $\sec\theta>0$. The limits: $x=0$ gives $\theta=\arctan 0=0$, and $x=\dfrac{4\sqrt{3}}{3}$ gives $\tan\theta=\dfrac{\sqrt{3}}{3}$, so $\theta=\dfrac{\pi}{6}$.

$$\int_0^{\frac{4\sqrt{3}}{3}} \frac{dx}{\left(x^2+16\right)^{5/2}}=\int_0^{\frac{\pi}{6}} \frac{4\sec^2\theta}{1{,}024\sec^5\theta}\,d\theta=\frac{1}{256}\int_0^{\frac{\pi}{6}}\cos^3\theta\,d\theta.$$

Write $\cos^3\theta=\left(1-\sin^2\theta\right)\cos\theta$ and let $u=\sin\theta$: $\displaystyle\int\cos^3\theta\,d\theta=\sin\theta-\frac{\sin^3\theta}{3}+C$. At $\theta=\dfrac{\pi}{6}$, $\sin\theta=\dfrac{1}{2}$, and at $\theta=0$ the bracket is 0:

$$\frac{1}{256}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{6}}=\frac{1}{256}\cdot\frac{11}{24}=\frac{11}{6144}.$$

No return to x is needed, because the limits moved with the variable.

7. Show that $\displaystyle\int_a^{2a}\frac{\sqrt{x^2-a^2}}{x^2}\,dx$ has the same value for every $a>0$, and find that value.

ANSWER

The value is $\ln\left(2+\sqrt{3}\right)-\dfrac{\sqrt{3}}{2}$ for every $a>0$ (about 0.451).

SOLUTION

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On $a \le x \le 2a$ let $x = a\sec\theta$ with $0 \le \theta < \dfrac{\pi}{2}$. Then $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2 - a^2} = a|\tan\theta| = a\tan\theta$, since $\tan\theta \ge 0$ there. The limits: $x = a$ gives $\sec\theta = 1$, so $\theta = 0$; $x = 2a$ gives $\sec\theta = 2$, so $\theta = \dfrac{\pi}{3}$. Neither depends on a.

$$\int_a^{2a} \frac{\sqrt{x^2 - a^2}}{x^2}\,dx = \int_0^{\pi/3} \frac{a\tan\theta \cdot a\sec\theta\tan\theta}{a^2\sec^2\theta}\,d\theta = \int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta}\,d\theta.$$

Every a has cancelled, and the limits are free of a, so the value is the same for every $a > 0$. To find it, $\dfrac{\tan^2\theta}{\sec\theta} = \dfrac{\sec^2\theta - 1}{\sec\theta} = \sec\theta - \cos\theta$:

$$\int_0^{\pi/3} (\sec\theta - \cos\theta)\,d\theta = \Big[\ln|\sec\theta + \tan\theta| - \sin\theta\Big]_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right) - \frac{\sqrt{3}}{2}.$$

At $\theta = 0$ the bracket is $\ln 1 - 0 = 0$. The value is about 0.451.

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