The ratio and root tests

Factorials and powers make terms hard to compare with anything, yet the ratio of one term to the next is often simple. When does that ratio, or the nth root of a term, decide whether a series converges?

WORDS TO KNOW

Ratio: $\left|\frac{a_{n+1}}{a_n}\right|$, the factor by which the size of a term changes at the next step.

Factorial: $n! = 1 \cdot 2 \cdots n$, so $(n+1)! = (n+1)\,n!$.

nth root: $\sqrt[n]{|a_n|}$, the rate r for which $|a_n|$ is roughly r^n.

Inconclusive: the test gives no answer, and you must use another test.

KEY IDEAS

1. The ratio test: compare with a geometric series

If the ratios settle at a number $L < 1$, then far enough out every ratio is below a fixed $r < 1$, and the terms are trapped under a geometric series with ratio r. If $L > 1$, the terms eventually grow and cannot tend to 0.

Ratio test. Suppose $a_n \neq 0$ from some index on and

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists, as a number or ∞. If $L < 1$, the series converges absolutely. If $L > 1$ or $L = \infty$, it diverges. If $L = 1$, the test gives no conclusion.

2. The root test: the average rate

Root test. Suppose $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, as a number or ∞. The conclusions are the same: absolute convergence if $L < 1$, divergence if $L > 1$ or $L = \infty$, nothing if $L = 1$.

Which one? The ratio test suits factorials, which cancel in $\frac{a_{n+1}}{a_n}$. The root test suits a whole term raised to the power n (or n^2).

3. A limit of 1 decides nothing

Every p-series $\sum \frac{1}{n^p}$ gives $L = 1$ in both tests, yet it converges for $p > 1$ and diverges for $p \leq 1$. So $L = 1$ sends you to another test, such as comparison.

Tail bound. If $\left|\frac{a_{n+1}}{a_n}\right| \leq r < 1$ for every $n \geq N+1$, then

$$|S - S_N| \leq \frac{|a_{N+1}|}{1-r}$$

WATCH OUT

- **Each ratio below 1 is not enough.** The harmonic series has ratios $\frac{n}{n+1} < 1$ and diverges. The proof needs one fixed $r < 1$ that bounds every ratio from some point on.
- **Rational terms always give $L = 1$.** For a term that is a quotient of polynomials, skip these tests and compare with a p-series.
- **These tests never show conditional convergence.** They give absolute convergence or divergence; a conditionally convergent series gives $L = 1$ or no limit.

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WORKED EXAMPLES

Example 1. Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

1. The terms are positive, and the factorials point to the ratio test.
2. Use $(n+1)! = (n+1)\,n!$ and $(2n+2)! = (2n+2)(2n+1)\,(2n)!$. Most factors cancel:
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2(2n+1)}$$
3. Divide top and bottom by n: the ratio tends to $L = \frac{1}{4}$, which is less than 1.

Answer: converges absolutely, by the ratio test with $L = \frac{1}{4}$. **Check:** the first terms are $\frac{1}{2}, \frac{1}{6}, \frac{1}{20}, \frac{1}{70}$, with ratios $\frac{1}{3}, \frac{3}{10}, \frac{2}{7}$ moving toward $\frac{1}{4}$.

Example 2. Decide whether $\displaystyle\sum_{n=2}^{\infty} \left(\frac{\ln n}{n}\right)^n$ converges.

1. For $n \geq 2$, $\ln n > 0$, so the terms are positive. The whole term is raised to the power n, so use the root test.
2. The nth root undoes the power: $\sqrt[n]{a_n} = \frac{\ln n}{n}$.
3. By L'Hôpital's rule, $\frac{\ln x}{x}$ has the same limit as $\frac{1/x}{1}$, which is 0. So $L = 0 < 1$.

Answer: converges absolutely, by the root test with $L = 0$. **Check:** the terms fall fast: about 0.120, 0.049, 0.014 for $n = 2, 3, 4$.

Example 3. Decide whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)}$ converges.

1. The ratio is
$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n}\,(n+1)}{\sqrt{n+1}\,(n+2)} = \sqrt{\frac{n}{n+1}} \cdot \frac{n+1}{n+2} \to 1 \cdot 1 = 1$$
so the ratio test says nothing.
2. The dominant part of the bottom is $\sqrt{n} \cdot n = n^{3/2}$, so use limit comparison with $b_n = \frac{1}{n^{3/2}}$:
$$\frac{a_n}{b_n} = \frac{n^{3/2}}{\sqrt{n}\,(n+1)} = \frac{n}{n+1} \to 1$$
3. The limit is positive and finite, and $\sum \frac{1}{n^{3/2}}$ converges ($p = \frac{3}{2}$).

Answer: converges, by limit comparison with $\sum \frac{1}{n^{3/2}}$; the ratio test was inconclusive.

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