Calculus 2 Unit 5 worksheet: The ratio and root tests (answer key)

ANSWERS AT A GLANCE

1. All by the ratio test. (a) $L = 0$, converges absolutely (b) $L = \frac{1}{3}$, converges absolutely (c) $L = \infty$, diverges (d) $L = \frac{2}{e}$, converges absolutely (e) $L = \frac{e^2}{7} > 1$, diverges
2. All by the root test. (a) $L = \frac{4}{3}$, diverges (b) $L = 0$, converges absolutely (c) $L = \frac{1}{2}$, converges absolutely (d) $L = \frac{5}{2}$, diverges
3. (a) Ratio limit 1; diverges by the limit comparison test with $\sum \frac{1}{n}$. (b) Ratio limit 1; converges by the alternating series test, but $\sum \frac{1}{\ln n}$ diverges by direct comparison with $\sum \frac{1}{n}$, so the convergence is conditional.
4. (a) $\frac{a_{n+1}}{a_n} = \frac{n+1}{n(n+2)} \to 0$; converges absolutely by the ratio test. (b) With $r = \frac{7}{48}$, $0 < S - S_5 \le \frac{2}{1435} \approx 0.001394$; $S_5 = \frac{719}{720}$. (c) $S = 1$; $S - S_5 = \frac{1}{720} \approx 0.001389$, which is $\frac{287}{288}$ of the bound.
5. (a) By the root test: $\sum a_n^2$ has $L = \frac{1}{9}$, converges; $\sum 2^n a_n$ has $L = \frac{2}{3}$, converges; $\sum 4^n a_n$ has $L = \frac{4}{3}$, diverges; $\sum n^5 a_n$ has $L = \frac{1}{3}$, converges. (b) $a_n = \frac{1}{n}$: $\sum \frac{1}{n^2}$ converges; $a_n = \frac{1}{\sqrt{n}}$: $\sum \frac{1}{n}$ diverges. A root limit of 1 for a_n gives no information about $\sum a_n^2$ either.
6. (a) The ratios alternate between $\frac{3}{4}$ (from odd n) and $\frac{1}{12}$ (from even n). (b) Every ratio is at most $\frac{3}{4} < 1$, so the series converges absolutely, eventually below a fixed rate. (c) $0 < S - S_4 \le \frac{1}{256}$; $S = \frac{7}{15}$, $S_4 = \frac{119}{256}$, true error $\frac{7}{3840}$, which is $\frac{7}{15}$ of the bound $\frac{15}{3840}$. (d) $L = \frac{1}{4} < 1$: converges absolutely.

WORKED SOLUTIONS

1. Use the ratio test to decide whether each series converges absolutely or diverges. Simplify $\left|\frac{a_{n+1}}{a_n}\right|$ before taking the limit L.

(a) $\sum_{n=1}^{\infty} \frac{n^2\, 3^n}{n!}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$

(c) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(d) $\sum_{n=1}^{\infty} \frac{(-2)^n\, n!}{n^n}$

(e) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^5\, 7^n}$

ANSWER

All by the ratio test. (a) $L = 0$, converges absolutely (b) $L = \frac{1}{3}$, converges absolutely (c) $L = \infty$, diverges (d) $L = \frac{2}{e}$, converges absolutely (e) $L = \frac{e^2}{7} > 1$, diverges

SOLUTION

(a) Using $(n+1)! = (n+1)\, n!$,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2\, 3^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2\, 3^n} = \frac{3(n+1)^2}{(n+1)\, n^2} = \frac{3(n+1)}{n^2} \to 0$$

So $L = 0 < 1$ and the series converges absolutely by the ratio test (its terms are positive, so absolute convergence is plain convergence).

(b) The denominator of a_{n+1} has one more factor, $3(n+1) - 2 = 3n + 1$, than that of a_n, so

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{1 \cdot 4 \cdots (3n-2)}{1 \cdot 4 \cdots (3n-2)(3n+1)} = \frac{n+1}{3n+1} \to \frac{1}{3}$$

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$L = \dfrac{1}{3} < 1$: the series converges absolutely by the ratio test.

(c) $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{100^{n+1}} \cdot \dfrac{100^n}{n!} = \dfrac{n+1}{100} \to \infty$. So $L = \infty$ and the series diverges by the ratio test. (The terms shrink until $n = 99$, then grow for good.)

(d) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{2^n\, n!} = \dfrac{2(n+1)\, n^n}{(n+1)^{n+1}} = \dfrac{2\, n^n}{(n+1)^n} = \dfrac{2}{\left(1+\frac{1}{n}\right)^n}$. Since $\left(1+\dfrac{1}{n}\right)^n \to e$, $L = \dfrac{2}{e} \approx 0.736 < 1$: the series converges absolutely by the ratio test.

(e) $\dfrac{a_{n+1}}{a_n} = \dfrac{e^{2n+2}}{(n+1)^5\, 7^{n+1}} \cdot \dfrac{n^5\, 7^n}{e^{2n}} = \dfrac{e^2}{7}\left(\dfrac{n}{n+1}\right)^5 \to \dfrac{e^2}{7}$. Since $e^2 \approx 7.389 > 7$, $L \approx 1.056 > 1$, and the series diverges by the ratio test, although its first terms shrink.

2. Use the root test to decide whether each series converges absolutely or diverges. Give the limit L of $\sqrt[n]{|a_n|}$. You may use $\sqrt[n]{n} \to 1$.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n-1}{3n+5}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\, n^4}{n^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - \frac{1}{2}\right)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n^2\, 2^{n+3}}$

ANSWER

All by the root test. (a) $L = \dfrac{4}{3}$, diverges (b) $L = 0$, converges absolutely (c) $L = \dfrac{1}{2}$, converges absolutely (d) $L = \dfrac{5}{2}$, diverges

SOLUTION

(a) The terms are positive and $\sqrt[n]{a_n} = \dfrac{4n-1}{3n+5} = \dfrac{4 - 1/n}{3 + 5/n} \to \dfrac{4}{3}$. $L = \dfrac{4}{3} > 1$: the series diverges by the root test.

(b) $\sqrt[n]{a_n} = \dfrac{3\left(\sqrt[n]{n}\right)^4}{n}$. The top tends to $3 \cdot 1^4 = 3$ and the bottom to ∞, so $L = 0 < 1$: the series converges absolutely by the root test.

(c) Since $n \geq 1$, $\sqrt[n]{n} \geq 1$, so the base is at least $\dfrac{1}{2}$ and the terms are positive. $\sqrt[n]{a_n} = \sqrt[n]{n} - \dfrac{1}{2} \to 1 - \dfrac{1}{2} = \dfrac{1}{2}$. $L = \dfrac{1}{2} < 1$: the series converges absolutely by the root test.

(d) $\sqrt[n]{|a_n|} = \dfrac{5}{\left(\sqrt[n]{n}\right)^2 \cdot 2^{(n+3)/n}} = \dfrac{5}{\left(\sqrt[n]{n}\right)^2 \cdot 2 \cdot 2^{3/n}}$. Since $\sqrt[n]{n} \to 1$ and $2^{3/n} \to 1$, $L = \dfrac{5}{2} > 1$, and the series diverges by the root test.

3. For each series, show that the ratio test gives $L = 1$. Then decide whether the series converges by another test, named, and say whether any convergence is absolute or conditional.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n\, 2^n + 1}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

ANSWER

(a) Ratio limit 1; diverges by the limit comparison test with $\sum \dfrac{1}{n}$. (b) Ratio limit 1; converges by the alternating series test, but $\sum \dfrac{1}{\ln n}$ diverges by direct comparison with $\sum \dfrac{1}{n}$, so the convergence is conditional.

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SOLUTION

(a) Dividing top and bottom by 2^{n+1},

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)2^{n+1}+1} \cdot \frac{n\,2^n+1}{2^n} = \frac{2(n\,2^n+1)}{(n+1)2^{n+1}+1} = \frac{n+2^{-n}}{(n+1)+2^{-n-1}} \to 1$$

The ratio test says nothing. The powers of 2 nearly cancel, so compare with $b_n = \dfrac{1}{n}$: both are positive and

$$\frac{a_n}{b_n} = \frac{n\,2^n}{n\,2^n+1} = \frac{1}{1+\frac{1}{n\,2^n}} \to 1$$

a positive finite limit. The harmonic series diverges, so the series diverges by the limit comparison test.

(b) $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{\ln n}{\ln(n+1)}$. Both logarithms tend to ∞; by L'Hopital's rule on $\dfrac{\ln x}{\ln(x+1)}$, the limit is that of $\dfrac{1/x}{1/(x+1)} = \dfrac{x+1}{x}$, which is 1. The ratio test says nothing.

Alternating series test: $b_n = \dfrac{1}{\ln n} > 0$ for $n \geq 2$, it decreases because $\ln n$ increases, and $b_n \to 0$ because $\ln n \to \infty$. So the series converges. For absolute convergence, $\ln n < n$ gives $\dfrac{1}{\ln n} > \dfrac{1}{n}$ for $n \geq 2$, and $\sum \dfrac{1}{n}$ diverges, so $\sum \dfrac{1}{\ln n}$ diverges by the direct comparison test. The series converges conditionally, which is why neither the ratio nor the root test could have settled it: both detect only absolute convergence.

4. Let S be the sum of the series below, and S_N its Nth partial sum.

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

(a) Find the ratio $\dfrac{a_{n+1}}{a_n}$ and its limit, and state what the ratio test concludes.

(b) Show that the ratios decrease as n grows. Use the ratio tail bound with $N = 5$ to bound $S - S_5$, and give S_5 as a fraction.

(c) Show that $\dfrac{n}{(n+1)!} = \dfrac{1}{n!} - \dfrac{1}{(n+1)!}$, and use it to find S and the true value of $S - S_5$. What fraction of the bound in (b) does the true error use?

ANSWER

(a) $\dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{n(n+2)} \to 0$; converges absolutely by the ratio test. (b) With $r = \dfrac{7}{48}$, $0 < S - S_5 \leq \dfrac{2}{1435} \approx 0.001394$; $S_5 = \dfrac{719}{720}$. (c) $S = 1$; $S - S_5 = \dfrac{1}{720} \approx 0.001389$, which is $\dfrac{287}{288}$ of the bound.

SOLUTION

(a) $\dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{(n+2)!} \cdot \dfrac{(n+1)!}{n} = \dfrac{n+1}{n(n+2)}$, using $(n+2)! = (n+2)(n+1)!$. The top has degree 1 and the bottom degree 2, so the ratio tends to 0. $L = 0 < 1$: the series converges absolutely by the ratio test.

(b) With $m = n+1$, the ratio is $\dfrac{m}{m^2-1} = \dfrac{1}{m-\frac{1}{m}}$, and $m - \dfrac{1}{m}$ increases with m, so the ratios decrease. For every $n \geq 6$ the ratio is at most its value at $n = 6$, $r = \dfrac{7}{6 \cdot 8} = \dfrac{7}{48}$. The tail bound with $N = 5$ needs

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the rate from $n = 6$ on, so

$$0 < S - S_5 \le \frac{a_6}{1 - \frac{7}{48}} = \frac{6}{5040} \cdot \frac{48}{41} = \frac{1}{840} \cdot \frac{48}{41} = \frac{2}{1435} \approx 0.001394$$

The first terms are $\frac{1}{2}, \frac{2}{6}, \frac{3}{24}, \frac{4}{120}, \frac{5}{720}$, and over 720: $S_5 = \frac{360 + 240 + 90 + 24 + 5}{720} = \frac{719}{720}$.
(c) $\frac{1}{n!} - \frac{1}{(n+1)!} = \frac{(n+1) - 1}{(n+1)!} = \frac{n}{(n+1)!}$. The partial sums telescope: $S_N = \frac{1}{1!} - \frac{1}{(N+1)!} = 1 - \frac{1}{(N+1)!} \to 1$, so $S = 1$. Then $S_5 = 1 - \frac{1}{720}$ (agreeing with (b)), and $S - S_5 = \frac{1}{720} \approx 0.001389$. As a share of the bound, $\frac{1}{720} \div \frac{2}{1435} = \frac{1435}{1440} = \frac{287}{288}$: the bound is almost exact, because the ratios fall fast, so the tail is close to its first term times $\frac{1}{1 - r}$.

5. Suppose $a_n > 0$ for every n and $\sqrt[n]{a_n} \to \frac{1}{3}$.

(a) Decide whether each series converges, naming the test: $\sum a_n^2$, $\sum 2^n a_n$, $\sum 4^n a_n$, $\sum n^5 a_n$.

(b) Now suppose instead that $\sqrt[n]{a_n} \to 1$. Give one such sequence a_n for which $\sum a_n^2$ converges and one for which it diverges. What does this show?

ANSWER

(a) By the root test: $\sum a_n^2$ has $L = \frac{1}{9}$, converges; $\sum 2^n a_n$ has $L = \frac{2}{3}$, converges; $\sum 4^n a_n$ has $L = \frac{4}{3}$, diverges; $\sum n^5 a_n$ has $L = \frac{1}{3}$, converges. (b) $a_n = \frac{1}{n}$: $\sum \frac{1}{n^2}$ converges; $a_n = \frac{1}{\sqrt{n}}$: $\sum \frac{1}{n}$ diverges. A root limit of 1 for a_n gives no information about $\sum a_n^2$ either.

SOLUTION

(a) Each new series has positive terms, so take nth roots.

$\sqrt[n]{a_n^2} = \left(\sqrt[n]{a_n}\right)^2 \to \frac{1}{9} < 1$: $\sum a_n^2$ converges by the root test.

$\sqrt[n]{2^n a_n} = 2\sqrt[n]{a_n} \to \frac{2}{3} < 1$: converges by the root test.

$\sqrt[n]{4^n a_n} = 4\sqrt[n]{a_n} \to \frac{4}{3} > 1$: diverges by the root test.

$\sqrt[n]{n^5 a_n} = \left(\sqrt[n]{n}\right)^5 \sqrt[n]{a_n} \to 1^5 \cdot \frac{1}{3} = \frac{1}{3} < 1$: converges by the root test. A power of n does not change the rate.

(b) For $a_n = \frac{1}{n}$, $\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} \to 1$, and $\sum a_n^2 = \sum \frac{1}{n^2}$ converges (p series, $p = 2 > 1$). For $a_n = \frac{1}{\sqrt{n}}$, $\sqrt[n]{a_n} = \left(\sqrt[n]{n}\right)^{-1/2} \to 1$, and $\sum a_n^2 = \sum \frac{1}{n}$ diverges ($p = 1$). Squaring turns a root limit of 1 into $1^2 = 1$, still the silent case, so another test has to decide, and the answer can go either way.

6. Consider the series below, with sum S and Nth partial sum S_N.

$$\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{4^n} = \frac{1}{4} + \frac{3}{16} + \frac{1}{64} + \frac{3}{256} + \cdots$$

(a) Show that $\frac{a_{n+1}}{a_n}$ has no limit, so the ratio test in its limit form does not apply.

(b) Show that the series still converges, and name the result you use.

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(c) Use the ratio tail bound with $N = 4$ to bound $S - S_4$. Then find S exactly and compare the true error with the bound.

(d) What does the root test give for this series?

ANSWER

(a) The ratios alternate between $\frac{3}{4}$ (from odd n) and $\frac{1}{12}$ (from even n). (b) Every ratio is at most $\frac{3}{4} < 1$, so the series converges absolutely, eventually below a fixed rate. (c) $0 < S - S_4 \le \frac{1}{256}$; $S = \frac{7}{15}$, $S_4 = \frac{119}{256}$, true error $\frac{7}{3840}$, which is $\frac{7}{15}$ of the bound $\frac{15}{3840}$. (d) $L = \frac{1}{4} < 1$: converges absolutely.

SOLUTION

(a) For odd n, $a_n = \frac{1}{4^n}$; for even n, $a_n = \frac{3}{4^n}$. From an odd n the ratio is $\frac{3}{4^{n+1}} \cdot 4^n = \frac{3}{4}$; from an even n it is $\frac{1}{4^{n+1}} \cdot \frac{4^n}{3} = \frac{1}{12}$. The ratios take the two values $\frac{3}{4}$ and $\frac{1}{12}$ in turn, so they have no limit.

(b) Every ratio is at most $r = \frac{3}{4} < 1$, for every $n \ge 1$. By the result "eventually below a fixed rate", the series converges absolutely; no limit is needed.

(c) The ratios are at most $\frac{3}{4}$ from $n = 5$ on, and $a_5 = \frac{1}{4^5} = \frac{1}{1024}$, so

$$0 < S - S_4 \le \frac{a_5}{1 - \frac{3}{4}} = \frac{4}{1024} = \frac{1}{256}$$

The terms are positive, so they may be split: $\sum_{n=1}^{\infty} \frac{2}{4^n} = \frac{2 \cdot \frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3}$ and $\sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{-\frac{1}{4}}{1 + \frac{1}{4}} = -\frac{1}{5}$, both geometric. So $S = \frac{2}{3} - \frac{1}{5} = \frac{7}{15}$. Over 256, $S_4 = \frac{64 + 48 + 4 + 3}{256} = \frac{119}{256}$, and

$$S - S_4 = \frac{7}{15} - \frac{119}{256} = \frac{1792 - 1785}{3840} = \frac{7}{3840} \approx 0.00182$$

against the bound $\frac{1}{256} = \frac{15}{3840} \approx 0.00391$. The true error is $\frac{7}{15}$ of the bound: the bound pretends every step shrinks by only $\frac{3}{4}$, while half the steps shrink by $\frac{1}{12}$.

(d) $\sqrt[n]{a_n} = \frac{\sqrt[n]{2 + (-1)^n}}{4}$, where the top is 1 or $\sqrt[n]{3}$, and $\sqrt[n]{3} \to 1$. So $L = \frac{1}{4} < 1$ and the root test gives absolute convergence. The root reads the average rate: $\sqrt{\frac{3}{4} \cdot \frac{1}{12}} = \frac{1}{4}$.

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