The common Maclaurin series

If a Taylor polynomial never stops, which series come out for e to the x, sine and cosine? And how do a few known series give series for many other functions, with no new derivatives?

WORDS TO KNOW

Taylor series of f centered at a: $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$, the Taylor polynomial with no last term.

Maclaurin series: a Taylor series centered at 0.

Infinite radius: the series converges for every real x.

Built from a known series: found by substituting, multiplying by a constant or a power of x, adding, multiplying two series, or differentiating or integrating term by term.

KEY IDEAS

1. Three series from repeating derivatives

Every derivative of e^x is e^x, so every derivative at 0 is 1 and $c_k = \frac{1}{k!}$. The derivatives of $\sin x$ at 0 run 0, 1, 0, −1 and repeat, and those of $\cos x$ run 1, 0, −1, 0. For e^x the ratio of consecutive terms is $\frac{|x|}{n+1} \to 0$, so the radius is infinite. The series for sine and cosine have infinite radius too.

Function	Series	Equals the function on
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$(-1, 1)$
e^x	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	all real x
$\sin x$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$	all real x
$\cos x$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$	all real x
$\ln(1+x)$	$\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^n}{n}$	$(-1, 1]$
$\arctan x$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1}$	$[-1, 1]$

2. One center, one series

Uniqueness. If $f(x) = \sum c_k (x-a)^k$ on an open interval around a, then $c_k = \frac{f^{(k)}(a)}{k!}$ for every k.

So any power series for f, however you found it, is its Taylor series. Read backward, $f^{(k)}(a) = k!\, c_k$.

3. Build new series from the table

Substitute $u = g(x)$ into a series: the new series equals the function exactly where $g(x)$ lies in the old set.

Multiply two series by collecting every product of terms whose powers add to n. This is guaranteed inside both open intervals of convergence.

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WATCH OUT

- **Convergence is not equality.** A Taylor series can converge for every x and still sum to something other than its function. Equality has to be shown.
- **Every derivative must exist at the center.** $\sqrt{x}$ has no Maclaurin series, since it is not differentiable at 0.
- **Each series holds only on its own set.** After a substitution, check where $g(x)$ lands, endpoints included.
- **Coefficients are not derivatives.** Multiply by $k!$ to get $f^{(k)}(a)$ from c_k.

WORKED EXAMPLES

Example 1. Find the Maclaurin series of $f(x) = xe^{-x^2}$, where it equals f, and $f^{(7)}(0)$ and $f^{(8)}(0)$.

1. The series for e^u holds for every real u. With $u = -x^2$, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ for every real x.
2. Multiply every term by x:
$$xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} = x - x^3 + \frac{x^5}{2} - \frac{x^7}{6} + \cdots$$
3. By uniqueness this is the Maclaurin series of f, so $f^{(k)}(0) = k!\, c_k$. The power x^7 is x^{2n+1} with $n = 3$, so $c_7 = -\frac{1}{3!} = -\frac{1}{6}$.
4. Only odd powers appear, so $c_8 = 0$.

Answer: the series above, for every real x. Then $f^{(7)}(0) = 7! \cdot \left(-\frac{1}{6}\right) = -840$ and $f^{(8)}(0) = 0$. **Check:** f is an odd function, so its even-order derivatives at 0 are 0.

Example 2. Find the Maclaurin series of $\cos x \arctan x$ through x^5, and say where the product is guaranteed to equal the function.

1. Write $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$.
2. Collect the products whose powers add to each n. Cosine has only even powers and arctangent only odd ones, so only odd powers appear.

Power	Products	Coefficient
x	$1 \cdot 1$	1
x^3	$1 \cdot \left(-\frac{1}{3}\right) + \left(-\frac{1}{2}\right) \cdot 1$	$-\frac{5}{6}$
x^5	$1 \cdot \frac{1}{5} + \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) + \frac{1}{24} \cdot 1$	$\frac{49}{120}$

Answer: $\cos x \arctan x = x - \frac{5}{6}x^3 + \frac{49}{120}x^5 - \cdots$, guaranteed for $-1 < x < 1$: the arctangent series has open interval of convergence $(-1, 1)$, and the cosine series converges everywhere. **Check:** for $f(x) = \cos x \arctan x$, the product rule gives $f'(x) = -\sin x \arctan x + \frac{\cos x}{1+x^2}$, so $f'(0) = 1$, which is $1! \cdot c_1$.

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