Building a truth table for a compound statement

A statement can join several letters with several connectives at once. How do you build its truth table, one column at a time, so you know exactly when it is true?

WORDS TO KNOW

Compound statement: a statement built from simpler ones with ¬ (not), ∧ (and) and ∨ (or).

Piece (sub-formula): any part of the statement that is a statement itself, such as $\neg Q$ or $P \vee \neg Q$.

Fixed order: the standard way to list the rows: T before F, with the first letter changing slowest.

Letter: a simple statement such as P, Q or R, true or false on its own.

Helper column: a column for one piece, built only so a later column can use it.

KEY IDEAS

1. The letters fix the rows

Each letter is T or F, whatever the others are. So n letters give 2^n rows: no fewer, or a case is missed, and no more, or a case repeats.

> Two letters: P reads T, T, F, F and Q reads T, F, T, F.
>
> Three letters: P reads T, T, T, T, F, F, F, F; Q reads T, T, F, F, T, T, F, F; R reads T, F, T, F, T, F, T, F.

Keep to the same order every time, so two tables can be compared row by row.

2. A column for every piece, from the inside out

Build whatever sits inside the most parentheses first. A ¬ column needs only the column it flips. An ∧ or ∨ column needs the two columns it joins. Fill each new column by looking sideways at columns already on the page, using the small table for one connective.

> ¬: flip each entry. ∧: T only where both inputs are T. ∨: F only where both inputs are F.
>
> A piece that appears twice gets one column, read twice. The last column is the answer.

WATCH OUT

- **A ¬ covers only what comes straight after it.** $\neg P \wedge Q$ means $(\neg P) \wedge Q$. To deny the whole "and", write $\neg(P \wedge Q)$. The two have different tables.
- **Mixing ∧ and ∨ needs parentheses.** $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ are different statements, so check which one the words mean.
- **Do not jump to the last column.** Guessing from the whole statement is where mistakes come from. Build each helper column, then read across.
- **Count rows from the letters the table is built on.** A table on P, Q, R has 8 rows even if the statement only uses P and Q.

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WORKED EXAMPLES

Example 1. Build a truth table for $\neg(P \wedge Q) \wedge Q$.

1. Two letters, so $2^2 = 4$ rows. Innermost piece: $P \wedge Q$, T only where both letters are T (row 1).
2. Next: $\neg(P \wedge Q)$, the $P \wedge Q$ column flipped.
3. Last: join that column with Q by $\wedge$, T only where both are T (row 3).

Row	P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg(P \wedge Q) \wedge Q$
1	T	T	T	F	F
2	T	F	F	T	F
3	F	T	F	T	T
4	F	F	F	T	F

Answer: the last column is F, F, T, F. It is true only when P is false and Q is true.

Example 2. Build a truth table for $\neg(Q \wedge \neg R) \wedge P$.

1. Three letters, so $2^3 = 8$ rows. Innermost: $\neg R$, the R column flipped.
2. Then $Q \wedge \neg R$: T only where Q is T and R is F (rows 2 and 6).
3. Then negate that whole column, giving $\neg(Q \wedge \neg R)$.
4. Last: join it with P by $\wedge$. Rows 5 to 8 have P false, so they are all F.

Row	P	Q	R	$\neg R$	$Q \wedge \neg R$	$\neg(Q \wedge \neg R)$	**last**
1	T	T	T	F	F	T	T
2	T	T	F	T	T	F	F
3	T	F	T	F	F	T	T
4	T	F	F	T	F	T	T
5	F	T	T	F	F	T	F
6	F	T	F	T	T	F	F
7	F	F	T	F	F	T	F
8	F	F	F	T	F	T	F

Answer: T, F, T, T, F, F, F, F. It is true in rows 1, 3 and 4.

Example 3. A phone unlocks when the face is recognized, or when the passcode is entered and lockdown mode is off. In which rows does it unlock?

1. Let P be "the face is recognized", Q be "the passcode is entered" and R be "lockdown mode is on". The comma puts the "and" inside, so the rule is $P \vee (Q \wedge \neg R)$.
2. P is T in rows 1 to 4. $Q \wedge \neg R$ needs Q true and R false: rows 2 and 6. The $\vee$ needs at least one, so take every row on either list.

Answer: $P \vee (Q \wedge \neg R)$, true in rows 1, 2, 3, 4 and 6. **Check:** row 8 has lockdown off but neither the face nor the passcode, so the phone stays locked.

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