A conditional written as an or

The statement "if P then Q" says exactly the same as "P is false, or Q is true". How does that rewriting work, and how does it show you the negation of a conditional?

WORDS TO KNOW

Conditional $P \to Q$: "if P, then Q", false only when P is true and Q is false.

Logically equivalent ($\equiv$): having the same truth value in every row of a truth table.

Counterexample: one case where the hypothesis is true and the conclusion is false.

Or-form: the statement $\neg P \vee Q$, "P is false, or Q is true".

Vacuously true: true because the hypothesis is false.

KEY IDEAS

1. A conditional is an or in disguise

A conditional breaks in only one way: P true and Q false. The "or" $\neg P \vee Q$ fails only when both parts fail: P true and Q false. Same single case, so the two columns match.

Row	P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$
1	T	T	F	T	T
2	T	F	F	F	F
3	F	T	T	T	T
4	F	F	T	T	T

$$P \to Q \;\equiv\; \neg P \vee Q$$

If P is false, then $\neg P$ is true, and the "or" is already true. That is why a false hypothesis makes a conditional true.

2. The negation of a conditional is an "and"

Rewrite as an "or", then use De Morgan's law and cancel the double negation:

$$\neg(P \to Q) \equiv \neg(\neg P \vee Q) \equiv \neg\neg P \wedge \neg Q \equiv P \wedge \neg Q$$

The negation is "P and not Q". It is never another conditional.

3. Breaking a claim about every x

"For every x, if P then Q" is false exactly when some single value of x makes P true and Q false. So the negation tells you what to search for: a value where the hypothesis holds and the conclusion fails.

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WATCH OUT

- **"If P, then not Q" is not the negation.** When P is false, both $P \to Q$ and $P \to \neg Q$ are vacuously true. A statement and its negation can never be true together.
- **Put the "not" on the hypothesis.** In $\neg P \lor Q$ the "not" sits on P, the "if" part. Putting it on the conclusion instead gives a statement that does not match $P \to Q$.
- **Negate inequalities exactly.** The or-form of "if $x > 5$, then $x^2 > 20$" is "$x \leq 5$, or $x^2 > 20$". Writing $x < 5$ would leave out $x = 5$.
- **Rewriting is not proving.** The or-form has the same truth value. You still have to show it is true, or find a counterexample.

WORKED EXAMPLES

Example 1. Rewrite "If a number is a multiple of 10, then it is even" without the word "if". Is it true?

1. P is "the number is a multiple of 10" and Q is "the number is even".
2. The or-form $\neg P \lor Q$ reads: "A number is not a multiple of 10, or it is even."
3. Every multiple of 10 is $10k = 2(5k)$ for an integer k, so it is even. The conditional is true for every whole number, and so is its or-form.

Answer: "A number is not a multiple of 10, or it is even." True for every whole number.

Example 2. Negate "If it is a holiday, the library is closed" without writing another conditional.

1. Let P be "it is a holiday" and Q be "the library is closed". The statement is $P \to Q$.
2. Its or-form is $\neg P \lor Q$: "it is not a holiday, or the library is closed".
3. Negate with De Morgan's law: $\neg(\neg P \lor Q) \equiv P \land \neg Q$.
4. The guess "if it is a holiday, the library is open" fails: on a day that is not a holiday, it and the original are both vacuously true.

Answer: "It is a holiday, and the library is open."

Example 3. Show that the claim "for every real number x, if $x^2 \geq x$, then $x \geq 1$" is false.

1. The claim fails at a value of x exactly when the negation $P \land \neg Q$ holds there: $x^2 \geq x$ and $x < 1$.
2. Try $x = 0$. Then $x^2 = 0$, and 0 ≥ 0 holds (the sign allows equal), so the hypothesis holds.
3. Also 0 < 1, so the conclusion $x \geq 1$ fails.

Answer: the claim is false, and $x = 0$ is a counterexample. **Check:** $0^2 = 0 \geq 0$, but 0 ≥ 1 is false.

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