And, or: conjunction and disjunction

Two claims are joined into one sentence by "and" or by "or". When is the whole sentence true, and how do you list every case so nothing is missed?

WORDS TO KNOW

Statement: a sentence that is either true (T) or false (F), such as "12 is even".

Disjunction $P \vee Q$: "P or Q". True when at least one part is true, both included.

$\neg P$**:** "not P". It has the opposite truth value to P.

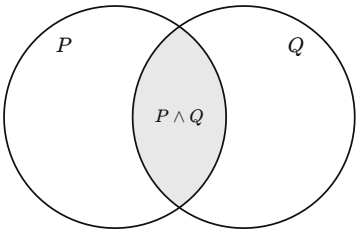
Conjunction $P \wedge Q$: "P and Q". True only when both parts are true.

Exclusive or $P \oplus Q$: "P or Q, but not both". True when exactly one part is true.

Truth table: a table with one row for each way of making the letters T or F, and a column showing the value of the statement in each row.

KEY IDEAS

1. "And" needs both parts



Think of P and Q as two circles of cases. $P \wedge Q$ holds only in the overlap, where both are true at once. Adding a second part to an "and" can only narrow it down.

> $P \wedge Q$ is true exactly when P is true and Q is true.
>
> To show an "and" is true, show every part. To show it is false, one false part is enough.

2. "Or" means at least one, and both counts

In math, "or" is **inclusive**: the overlap, where both parts are true, counts. The reading "one but not both" is a different connective, exclusive or. The two differ only in row 1.

Row	P	Q	$P \wedge Q$	$P \vee Q$	$P \oplus Q$
1	T	T	T	T	F
2	T	F	F	T	T
3	F	T	F	T	T
4	F	F	F	F	F

> $P \vee Q$ is false exactly when P and Q are both false.
>
> So an "and" column has exactly one T, and an "or" column has exactly one F.

3. Every row, in a fixed order

Each letter is T or F, so 2 letters give 2 × 2 = 4 rows, and n letters give 2^n rows. The usual order puts T before F and lets the first letter change slowest. Only the truth values of the parts matter, never what they say.

> With n letters: $P_1 \wedge \cdots \wedge P_n$ is true in exactly 1 row (all T), and $P_1 \vee \cdots \vee P_n$ is true in $2^n - 1$ rows (all but the all-F row).

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WATCH OUT

- **"Or" includes both.** "5 > 2 or 5 is odd" is true, even though both parts are true. Only "but not both" rules out row 1.
- **A not belongs to what it touches.** $\neg P \vee Q$ means "(not P) or Q". It is not "not (P or Q)", which is a different statement.
- **A false "and" does not say which part failed,** and a true "or" does not say which part holds. You need one more fact to pin down the other part.
- **Strict signs.** "$x < 4$" is false at $x = 4$, since 4 is not less than itself. Check the end points of an inequality one at a time.

WORKED EXAMPLES

Example 1. Decide whether each statement is true or false, and name its row of the four-row table. (a) 18 is even and 18 is a multiple of 4. (b) 11 is even or 11 is prime. (c) 3 > 8 or 3 is even.

1. Settle each part first. Then find the row with that pair of values.
2. (a) 18 = 2 · 9 is even, but 18 ÷ 4 = 4.5 is not whole. The pair is T, F: row 2. An "and" with a false part is false.
3. (b) 11 is odd, so the first part is false, but 11 is prime. The pair is F, T: row 3. One true part makes an "or" true.
4. (c) 3 > 8 is false, and 3 is odd. The pair is F, F: row 4, the only row where an "or" is false.

Answer: (a) false, row 2. (b) true, row 3. (c) false, row 4.

Example 2. For which of $x = -3, -1, 1, 4$ is "$x < -1$ or $x \geq 4$" true?

1. Each value of x turns the sentence into a statement. Test each value in both parts, as in the table below.
2. For $x = -1$, the part −1 < −1 is false, since −1 is not less than itself.
3. For $x = 4$, the part 4 ≥ 4 is true, since 4 = 4.

x	$x < -1$	$x \geq 4$	**either**
−3	T	F	T
−1	F	F	F
1	F	F	F
4	F	T	T

Answer: $x = -3$ and $x = 4$.

Example 3. The sentence "$x \leq -4$ or $x > 1$" is false. What can you say about x?

1. An "or" is false only in row 4, where both parts are false.
2. "$x \leq -4$" is false, so $x > -4$.
3. "$x > 1$" is false, so $x \leq 1$.
4. Both must hold at once, which is an "and".

Answer: $-4 < x \leq 1$. **Check:** $x = 0$ makes both parts false; $x = -4$ makes the first part true, so it is left out.

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