The contrapositive

Turning "if P, then Q" around can break a true statement. Which way of turning it around is always safe, and how do you use it to decide or prove a claim?

WORDS TO KNOW

Conditional $P \to Q$: "if P, then Q". P is the hypothesis, Q the conclusion.

Converse $Q \to P$: swap the two parts.

Inverse $\neg P \to \neg Q$: negate both parts.

Contrapositive $\neg Q \to \neg P$: swap the two parts and negate both.

Logically equivalent ($\equiv$): true together and false together, in every row of a truth table.

KEY IDEAS

1. Each conditional rules out one case

A conditional is false only when its hypothesis is true and its conclusion is false. Read off the one case each form rules out:

Statement	It is false only when
$P \to Q$	P is true and Q is false
$\neg Q \to \neg P$	$\neg Q$ is true and $\neg P$ is false: P true, Q false
$Q \to P$	Q is true and P is false
$\neg P \to \neg Q$	$\neg P$ is true and $\neg Q$ is false: P false, Q true

The original and the contrapositive rule out the same case, so they say the same thing. The converse and the inverse share a different case.

$$P \to Q \;\equiv\; \neg Q \to \neg P \qquad\qquad Q \to P \;\equiv\; \neg P \to \neg Q$$

But $P \to Q$ is **not** equivalent to its converse $Q \to P$. They disagree when exactly one of P, Q is true.

2. Use the contrapositive when it is easier

To decide or prove $P \to Q$, you may decide or prove $\neg Q \to \neg P$ instead. The answer is the same. This helps when $\neg Q$ gives you something concrete to start from.

To negate the parts: $x > 4$ becomes $x \leq 4$. "A or B" becomes "not A and not B". "Not odd" becomes "even". A claim "for every x" keeps its "for every"; only the two parts are negated.

WATCH OUT

- **The converse is not safe.** "If $x > 4$, then $x^2 > 16$" is true, but its converse fails at $x = -5$.
- **The contrapositive is not the negation.** Swapping and negating the parts keeps the meaning. Negating the whole statement reverses it.
- **Negate inequalities exactly.** The opposite of $x > 4$ is $x \leq 4$, not $x < 4$, because $x = 4$ must land on one side.
- **One counterexample breaks a pair.** A value that breaks the converse also breaks the inverse, since they rule out the same case.

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WORKED EXAMPLES

Example 1. For real x, take "if $x < -3$, then $x^2 > 9$." Write the converse, inverse and contrapositive, and decide which of the four are true.

1. P is $x < -3$ and Q is $x^2 > 9$. So $\neg P$ is $x \geq -3$ and $\neg Q$ is $x^2 \leq 9$.
2. Original: if $x < -3$, then $x^2 > 9$. True, since $-x > 3$ gives $x^2 = (-x)^2 > 9$.
3. Converse: if $x^2 > 9$, then $x < -3$. False: $x = 4$ gives 16 > 9, but 4 ≥ −3.
4. Inverse: if $x \geq -3$, then $x^2 \leq 9$. False: $x = 4$ again.
5. Contrapositive: if $x^2 \leq 9$, then $x \geq -3$. True, since $x^2 \leq 9$ means $-3 \leq x \leq 3$.

Answer: the original and the contrapositive are true; the converse and the inverse are false, both broken by $x = 4$.

Example 2. For integers a and b: "if $a + b$ is odd, then a is odd or b is odd." Decide whether it is true by using the contrapositive.

1. The negation of "$a + b$ is odd" is "$a + b$ is even".
2. The negation of "a is odd or b is odd" is "a is even and b is even", since an "or" is false only when both parts are.
3. Contrapositive: if a and b are both even, then $a + b$ is even.
4. Write $a = 2j$ and $b = 2k$. Then $a + b = 2(j + k)$, which is even. So the contrapositive is true.

Answer: true, because it is equivalent to its contrapositive, which is true.

Example 3. Prove that for every integer n: if $3n + 5$ is even, then n is odd.

Plan: starting from "$3n + 5$ is even" says little about n, so prove the contrapositive instead.

1. Contrapositive: if n is not odd, then $3n + 5$ is not even. That is: if n is even, then $3n + 5$ is odd.
2. Suppose n is even, so $n = 2k$ for an integer k.
3. Then $3n + 5 = 6k + 5 = 2(3k + 2) + 1$, which is one more than an even number.
4. So $3n + 5$ is odd. The contrapositive holds for every integer, so the original does too.

Answer: proved. **Check:** $n = 3$ gives 14, even, and 3 is odd. $n = 2$ gives 11, odd, so the statement makes no claim there.

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