De Morgan's laws in logic

How do you write the opposite of an "and" statement or an "or" statement? Negating does not just put a "not" on each part, because the connective flips too.

WORDS TO KNOW

Negation $\neg P$: "not P", true exactly when P is false.

De Morgan's laws: the two rules for negating an "and" and an "or".

$\equiv$ **(logically equivalent):** the two sides are true in exactly the same rows of a truth table.

Double negation: $\neg(\neg P) \equiv P$. Two nots cancel.

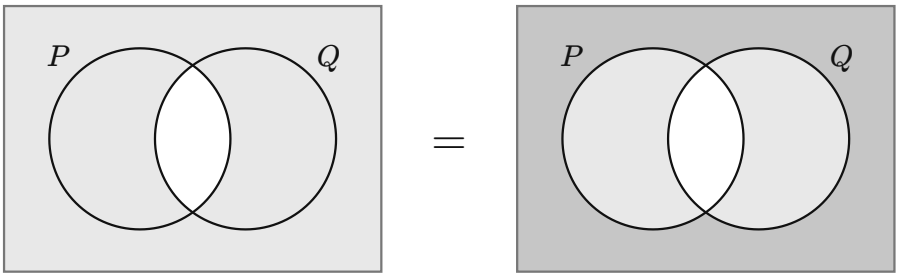
KEY IDEAS

1. The not moves inside, and the connective flips

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q \qquad \neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

The negation of an "and" is the "or" of the negations. The negation of an "or" is the "and" of the negations.

An "and" is easy to break: one failing part is enough. So its negation must be easy to satisfy, which is the mark of an "or". In the picture, "not (P and Q)" is everything outside the overlap, and that is exactly where you are outside P, or outside Q, or both.



$\neg(P \wedge Q)$ $\qquad$ $\neg P \vee \neg Q$

2. The table shows why the connective must flip

Row	P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg P \wedge \neg Q$
1	T	T	F	F	F
2	T	F	T	T	F
3	F	T	T	T	F
4	F	F	T	T	T

The columns for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ agree in every row. The guess $\neg P \wedge \neg Q$, which keeps the "and", fails in rows 2 and 3.

3. Longer and nested statements

Apply a law to one connective at a time, starting with the outside one. A chain of three or more parts needs the law once for each extra part.

$$\neg(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \equiv \neg P_1 \vee \neg P_2 \vee \cdots \vee \neg P_n$$

The same holds with $\wedge$ and $\vee$ swapped. Negating $x \leq a$ gives $x > a$, and negating $x < a$ gives $x \geq a$.

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WATCH OUT

- **Do not keep the connective.** $\neg(P \wedge Q)$ is not $\neg P \wedge \neg Q$. "Not (raining and cold)" is true on a day that is raining but warm, while "not raining and not cold" is false.
- **Strict and non-strict signs swap.** The negation of $x \leq 4$ is $x > 4$, not $x \geq 4$. The boundary point must change sides.
- **One law reaches one connective.** If a part is itself an "and" or an "or", apply a law again inside it.
- **These laws cover only $\wedge$, $\vee$ and $\neg$.** An "if...then" needs its own rule first.

WORKED EXAMPLES

Example 1. Negate "27 is odd and 27 is a multiple of 6." Decide whether each statement is true.

1. The original: 27 is odd, true. 27 ÷ 6 = 4.5, so 27 is not a multiple of 6: false. One false part makes the "and" false.
2. By the "and" law, the negation is "27 is even or 27 is not a multiple of 6."
3. The first part is false, but the second is true. One true part makes the "or" true.

Answer: the original is false and its negation is true, as they must be opposite.

Example 2. Negate "$x < -3$ or $x \geq 2$."

1. By the "or" law, the negation is the "and" of the two negations.
2. The negation of $x < -3$ is $x \geq -3$. The negation of $x \geq 2$ is $x < 2$.

Answer: $x \geq -3$ and $x < 2$, that is, $-3 \leq x < 2$. **Check:** at $x = 2$ the original is true (2 ≥ 2) and the negation is false (2 < 2 fails). At $x = -3$ the original is false and the negation is true.

Example 3. Negate "(n is odd and $n > 5$) or n is a multiple of 4." Evaluate both statements at $n = 3$.

1. Name the parts: A is "n is odd", B is "$n > 5$", C is "n is a multiple of 4". The statement is $(A \wedge B) \vee C$.
2. The "or" law on the outside, then the "and" law on the part it exposes:
$$\neg\big((A \wedge B) \vee C\big) \equiv \neg(A \wedge B) \wedge \neg C \equiv (\neg A \vee \neg B) \wedge \neg C$$
3. In words: "(n is even or $n \leq 5$) and n is not a multiple of 4."
4. At $n = 3$: 3 > 5 is false, so $A \wedge B$ is false, and 3 is not a multiple of 4. The original is false.
5. The negation: 3 ≤ 5 is true, so the inner "or" is true, and 3 is not a multiple of 4. Both parts hold, so it is true.

Answer: "(n is even or $n \leq 5$) and n is not a multiple of 4." At $n = 3$ the original is false and the negation is true.

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