Tautologies and contradictions

Some statements come out true in every row of their truth table, and some come out false in every row. How do you tell, and why does it matter?

WORDS TO KNOW

Tautology: a statement that is T in every row of its truth table, whatever its letters stand for.

Contingent (neither): T in some rows and F in others. Most statements are this kind.

Valid argument: no row makes every premise true and the conclusion false.

Contradiction: a statement that is F in every row of its truth table.

Contradictory pair: two statements with opposite values in every row, such as P and $\neg P$.

KEY IDEAS

1. Three kinds, decided by the whole column

Build the table and read the last column. All T: tautology. All F: contradiction. A mix: neither. The verdict comes from how the connectives are put together, not from what the letters say.

Row	P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
1	T	F	T	F
2	F	T	T	F

$P \vee \neg P$ is a tautology and $P \wedge \neg P$ is a contradiction.

S is a contradiction exactly when $\neg S$ is a tautology, since $\neg$ flips every entry.

2. Without the full table: hunt for the bad row

To show a statement is a tautology, try to make it false. To show it is a contradiction, try to make it true. If every attempt forces some letter to be both T and F, no such row exists.

A conditional is false only with its hypothesis T and its conclusion F, so start there. An "and" is true only when every part is T.

3. What a tautology tests

An argument with premises $P_1, \ldots, P_n$ and conclusion C is valid exactly when

$$(P_1 \wedge \cdots \wedge P_n) \to C \text{ is a tautology.}$$

Two statements A and B are logically equivalent exactly when $A \leftrightarrow B$ is a tautology.

WATCH OUT

- **Every row counts.** A few T rows prove nothing. One F row is enough to show a statement is not a tautology.
- **"Not a tautology" does not mean "contradiction".** A statement with some T rows and some F rows is neither.
- **A tautology tells you nothing about its letters.** $P \vee \neg P$ is true whatever P stands for, so it cannot tell you whether that P is true.
- **A table sees only the logical form.** Two different letters can both be T in some row, even when the sentences they stand for can never both be true. Only a statement that is F in every row of its own table is a contradiction.

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WORKED EXAMPLES

Example 1. Is $(P \wedge \neg Q) \to \neg(P \wedge Q)$ a tautology?

1. Two letters, so 4 rows. Build $\neg Q$, then $P \wedge \neg Q$; separately $P \wedge Q$, then $\neg(P \wedge Q)$.
2. Apply the rule for $\to$: false only where the hypothesis is T and the conclusion is F.
3. The hypothesis $P \wedge \neg Q$ is T only in row 2, and the conclusion is T there.
4. In row 1 the conclusion is F, but the hypothesis is F too, so the row is T.

Row	P	Q	$\neg Q$	$P \wedge \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	**whole**
1	T	T	F	F	T	F	T
2	T	F	T	T	F	T	T
3	F	T	F	F	F	T	T
4	F	F	T	F	F	T	T

Answer: every row ends in T, so it is a tautology.

Example 2. Classify (a) $(P \to Q) \to \neg P$ and (b) $\neg(P \to Q) \wedge Q$.

1. (a) $P \to Q$ is T, F, T, T and $\neg P$ is F, F, T, T. The outer arrow fails only where its hypothesis is T and $\neg P$ is F: row 1. So the column is F, T, T, T.
2. (b) $\neg(P \to Q)$ is T only in row 2, where $P \to Q$ fails. In row 2, Q is F. So the "and" never has both parts T, and the column is F, F, F, F.

Answer: (a) neither: false in row 1, true in rows 2, 3 and 4. (b) a contradiction.

Example 3. Without writing all eight rows, show that $\big(P \to (Q \wedge R)\big) \to (P \to R)$ is a tautology.

1. Try to make it false. The conclusion $P \to R$ must be F, so P is T and R is F.
2. The hypothesis $P \to (Q \wedge R)$ must be T. Since P is T, that forces $Q \wedge R$ to be T.
3. Then R is T. But step 1 needed R to be F.

Answer: no row makes it false, so it is a tautology. In words: if P gives both Q and R, then P gives R.

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