The truth table for the conditional

When is "if P, then Q" true? Its truth table answers this in four rows, including the surprising rows where P is false.

WORDS TO KNOW

Conditional $P \to Q$: the statement "if P, then Q".

Conclusion: the "then" part, Q.

Converse: the conditional with the parts swapped, $Q \to P$.

Hypothesis: the "if" part, P.

Vacuously true: true because the hypothesis is false, so the statement makes no demand.

KEY IDEAS

1. Only one row breaks the promise

"If P, then Q" promises that Q holds whenever P holds. The promise is broken only when P holds and Q fails: row 2.

Row	P	Q	$P \to Q$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

$P \to Q$ is false exactly when P is true and Q is false. In every other row it is true.

2. A false hypothesis makes the conditional true

When P is false, the promise asks for nothing, so it cannot be broken. Here is why this choice is forced. "If n is a multiple of 10, then n is even" is true for every whole number n. At $n = 2$ it sits in row 3 (F, T), and at $n = 3$ in row 4 (F, F). If either row held F, this true claim would fail.

If the hypothesis is false, $P \to Q$ is true, whatever Q says. If the conclusion is true, $P \to Q$ is also true, whatever P says.

3. Inside bigger statements

To find where a longer conditional is false, find the rows where its hypothesis is true and its conclusion is false. Nothing else can make it false.

The grouping of arrows matters: $P \to (Q \to R)$ and $(P \to Q) \to R$ are different statements. The order matters too: the converse $Q \to P$ is a different statement from $P \to Q$.

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WATCH OUT

- **A conditional is not a claim of cause or connection.** "If 12 is prime, then 2 + 2 = 5" is true, since 12 is not prime. The table only asks: is the hypothesis true and the conclusion false?
- **A false hypothesis does not make the statement meaningless.** It makes the statement true.
- **The converse is a different promise.** "If you live in Paris, you live in France" is true, but its converse fails for someone who lives in Lyon.
- **Always write the parentheses in a chain of arrows.** Unlike "and" and "or", the grouping changes the meaning.

WORKED EXAMPLES

Example 1. A post office sign reads: "If a parcel weighs over 2 kilograms, an extra fee is charged." Make the truth table for the sign and say what each row means.

1. Let P be "the parcel weighs over 2 kilograms" and Q be "an extra fee is charged". The sign says $P \to Q$.
2. Row 1 (heavy, fee): the sign told the truth. Row 2 (heavy, no fee): the sign was broken.
3. Row 3 (light, fee anyway, perhaps for fast delivery): the sign said nothing about light parcels, so it was not broken. Row 4 (light, no fee): also true.

Row	P (over 2 kg)	Q (fee)	$P \to Q$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

Answer: only row 2, a heavy parcel with no fee, breaks the sign.

Example 2. Decide true or false. (a) If 21 is even, then 21 is prime. (b) If 21 is a multiple of 7, then 21 is even. (c) If 21 is prime, then 21 is odd.

1. Settle the parts. 21 is odd, 21 = 3 × 7 is a multiple of 7, and 21 is not prime.
2. (a) Hypothesis false, conclusion false: row 4, so the statement is vacuously true.
3. (b) Hypothesis true, conclusion false: row 2, the one false row.
4. (c) Hypothesis false, conclusion true: row 3, so the statement is true.

Answer: (a) true, (b) false, (c) true.

Example 3. In which rows of the table (letters P, Q, R in the fixed order) is $\neg(P \wedge Q) \to R$ false?

1. The hypothesis is $\neg(P \wedge Q)$. $P \wedge Q$ is true only in rows 1 and 2, so the hypothesis is true in rows 3 to 8.
2. The conclusion is R. It is false in rows 2, 4, 6 and 8.
3. The conditional is false only where both happen: the hypothesis true and the conclusion false.

Answer: rows 4, 6 and 8. It is true in the other five rows. **Check:** row 2 has R false, but there P and Q are both true, so the hypothesis is false and the row is vacuously true.

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