Discrete Mathematics and Proof Unit 1 practice: Tautologies and contradictions (answer key)

ANSWERS AT A GLANCE

1. (a) T, T: tautology (b) F, F: contradiction (c) T, T: tautology (d) F, F: contradiction (e) T, T: tautology (f) T, T: tautology
2. (a) tautology (b) neither: false in row 2 (c) tautology (d) contradiction (e) tautology (f) contradiction
3. (a) none: a tautology (b) $\big((P \to Q) \wedge (Q \to P)\big) \to P$; invalid, false in row 4 (c) not equivalent: they disagree in row 2
4. (a) $(K \wedge W) \to W$: a tautology (b) $\big((A \leftrightarrow D) \wedge \neg A\big) \to \neg D$; valid (c) 8

WORKED SOLUTIONS

1. Fill in the two rows of each statement's table (P = T, then P = F). Then say whether it is a tautology, a contradiction or neither.

(a) $P \leftrightarrow P$ (b) $\neg(P \vee P) \wedge P$ (c) $(P \to \neg P) \vee P$

(d) $(P \vee \neg P) \to (P \wedge \neg P)$ (e) $(P \to P) \vee \neg P$ (f) $P \leftrightarrow \neg\neg P$

ANSWER

(a) T, T: tautology (b) F, F: contradiction (c) T, T: tautology (d) F, F: contradiction (e) T, T: tautology (f) T, T: tautology

SOLUTION

A tautology is T on both rows, a contradiction is F on both rows, and anything else is neither.

(a) The two sides are the same letter, so they agree on both rows; $P \leftrightarrow P$: T, T, so tautology.

(b) $\neg(P \vee P)$: F, T; $\neg(P \vee P) \wedge P$: F, F, so contradiction.

(c) $P \to \neg P$: F, T; $(P \to \neg P) \vee P$: T, T, so tautology.

(d) $P \vee \neg P$: T, T; $P \wedge \neg P$: F, F; $(P \vee \neg P) \to (P \wedge \neg P)$: F, F, so contradiction.

(e) $P \to P$: T, T; $(P \to P) \vee \neg P$: T, T, so tautology.

(f) $\neg\neg P$: T, F; $P \leftrightarrow \neg\neg P$: T, T, so tautology.

2. Classify each statement as a tautology, a contradiction or neither (rows P, Q = TT, TF, FT, FF). When it is neither, name the rows where it is false.

(a) $(P \vee Q) \vee \neg P$ (b) $\neg P \vee (P \wedge Q)$ (c) $\neg Q \vee (P \to Q)$

(d) $\neg(P \vee \neg Q) \wedge \neg Q$ (e) $\neg(P \wedge Q) \vee P$ (f) $\neg\big(Q \to (P \vee Q)\big)$

ANSWER

(a) tautology (b) neither: false in row 2 (c) tautology (d) contradiction (e) tautology (f) contradiction

SOLUTION

Fill a helper column for each inner piece, then the last column.

(a) $P \vee Q$: T, T, T, F; $\neg P$: F, F, T, T; $(P \vee Q) \vee \neg P$: T, T, T, T, so tautology.

(b) $\neg P$: F, F, T, T; $P \wedge Q$: T, F, F, F; $\neg P \vee (P \wedge Q)$: T, F, T, T, so neither: false in row 2.

(c) $\neg Q$: F, T, F, T; $P \to Q$: T, F, T, T; $\neg Q \vee (P \to Q)$: T, T, T, T, so tautology.

(d) $\neg(P \vee \neg Q)$: F, F, T, F; $\neg Q$: F, T, F, T; $\neg(P \vee \neg Q) \wedge \neg Q$: F, F, F, F, so contradiction.

(e) $\neg(P \wedge Q)$: F, T, T, T; $\neg(P \wedge Q) \vee P$: T, T, T, T, so tautology.

(f) $Q \to (P \vee Q)$: T, T, T, T; $\neg\big(Q \to (P \vee Q)\big)$: F, F, F, F, so contradiction.

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3. Each of these takes two steps. Rows run in the fixed order: P, Q = TT, TF, FT, FF, and P, Q, R = TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF.

(a) Name every row where $\big(\neg R \wedge (P \to R)\big) \to \neg(P \wedge Q)$ is false, or say there is none.

(b) Premises $P \to Q$ and $Q \to P$, conclusion P. Write the argument as one conditional and decide whether it is valid.

(c) Use the biconditional test: are $\neg P \to (P \vee Q)$ and Q logically equivalent? If not, name the rows where they disagree.

ANSWER

(a) none: a tautology (b) $\big((P \to Q) \wedge (Q \to P)\big) \to P$; invalid, false in row 4 (c) not equivalent: they disagree in row 2

SOLUTION

(a) A false row needs $\neg(P \wedge Q)$ false, so P and Q true, and the hypothesis true, so R false. Then $P \to R$ is false and the hypothesis fails. No row is false: a tautology.

(b) The premises, joined by "and", go before the arrow: $\big((P \to Q) \wedge (Q \to P)\big) \to P$. A counterexample needs P false; then $Q \to P$ forces Q false, and $P \to Q$ is true. So row 4 is a counterexample: invalid.

(c) $\neg P \to (P \vee Q)$: T, T, T, F against Q: T, F, T, F. The biconditional is F in row 2, so it is not a tautology.

4. (a) Hannah says: "If the kettle is on and the water is hot, then the water is hot." Using K for "the kettle is on" and W for "the water is hot", write the sentence in symbols and say whether it is a tautology, a contradiction or neither.

(b) A friend reasons: "The alarm sounds if and only if the door is open. The alarm is silent. So the door is closed." Using A for "the alarm sounds" and D for "the door is open", write the argument as one conditional and decide whether it is valid.

(c) Hannah builds the truth table of a contradiction S that uses 3 letters, and adds a column for $\neg S$. How many entries of the $\neg S$ column are T?

ANSWER

(a) $(K \wedge W) \to W$: a tautology (b) $\big((A \leftrightarrow D) \wedge \neg A\big) \to \neg D$; valid (c) 8

SOLUTION

(a) $(K \wedge W) \to W$. The hypothesis $K \wedge W$ is T only when both letters are T, and then W is T too, so no row is false. So it is a tautology.

(b) $\big((A \leftrightarrow D) \wedge \neg A\big) \to \neg D$. A false row needs D true; then $A \leftrightarrow D$ forces A true, against $\neg A$. No row is false, so the conditional is a tautology. So the argument is valid.

(c) The table has $2^3 = 8$ rows. S is F on all of them, and $\neg S$ flips every entry, so all 8 entries are T.

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