Discrete Mathematics and Proof Unit 1 quiz 2: Tautologies and contradictions, De Morgan's laws for statements, A conditional and its contrapositive (answer key)

8 questions, 21 points

ANSWERS AT A GLANCE

1. Contradiction.
2. (a) when the student is not wearing safety glasses or no teacher is in the room (or both) (b) the first part of the "or" in (a) is true, and one true part makes an "or" true, so the student may not use the cutter
3. (a) the number is not a multiple of 2 and not a multiple of 5 (b) 10 tickets
4. (a) contradictory (b) not contradictory: the same value in rows 2 and 3 (c) contradictory
5. (a) P false, Q true (b) P true, Q false (c) R false and at least one of P, Q true: rows 2, 4 and 6
6. (a) $\big((D \to W) \wedge (W \to B) \wedge \neg B\big) \to \neg D$; valid (b) $\big((D \to W) \wedge (W \to B) \wedge \neg D\big) \to \neg B$; invalid, counterexamples: rows 5 and 7
7. (a) contrapositive (b) converse (c) inverse (d) none of these Of the four, only (a) is logically equivalent to $P \to \neg Q$.
8. (a) and (d) say the same as the rule. (b) and (c) do not: a visitor aged 12 or older who is with an adult keeps the rule but breaks both.

WORKED SOLUTIONS

1. Classify $(P \wedge Q) \wedge \neg(P \vee Q)$ as a tautology, a contradiction or neither, using a truth table with the letters P, Q in the fixed order. If it is neither, name every row where it is false. (2 points)

   ANSWER
   Contradiction.

   SOLUTION
   Build the table with a column for each part of $(P \wedge Q) \wedge \neg(P \vee Q)$. The last column, rows 1 to 4, is F, F, F, F. Contradiction.

   SCORING
   1 point: last column F, F, F, F. 1 point: Contradiction.

2. A school lab allows a student to use the laser cutter exactly when the student is wearing safety glasses and a teacher is in the room. (2 points)

   (a) Use a De Morgan law to say, in words, exactly when a student may not use the laser cutter.

   (b) A student who is not wearing safety glasses says: "A teacher is in the room, so I may use the cutter." Use your answer to (a) to explain why not.

   ANSWER
   (a) when the student is not wearing safety glasses or no teacher is in the room (or both) (b) the first part of the "or" in (a) is true, and one true part makes an "or" true, so the student may not use the cutter

   SOLUTION
   (a) Use is allowed when $G \wedge T$, where G is "the student is wearing safety glasses" and T is "a teacher is in the room". Use is not allowed when $\neg(G \wedge T)$, which by the "and" law is $\neg G \vee \neg T$: the student is not wearing safety glasses, or no teacher is in the room, or both.

   (b) For this student $\neg G$ is true. One true part makes the "or" $\neg G \vee \neg T$ true, whatever T is, so the student may not use the cutter. A teacher being present makes $\neg T$ false, but an "or" needs only one true part.

   SCORING
   1 point: (a) the "or" of the two negations, not an "and". 1 point: (b) the missing glasses alone make the "or" true, so the teacher's presence does not matter.

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3. Tickets in a school raffle are numbered 1 to 24. A ticket wins a small prize unless its number is a multiple of 2 or a multiple of 5. (2 points)

(a) Use a De Morgan law to write the winning condition as an "and" of two conditions, with no "not" in front of a bracket.

(b) How many tickets win a small prize?

ANSWER

(a) the number is not a multiple of 2 and not a multiple of 5 (b) 10 tickets

SOLUTION

(a) Let A be "the number is a multiple of 2" and B be "the number is a multiple of 5". A ticket wins when $A \vee B$ is false, that is, when $\neg(A \vee B)$ is true. The "or" law gives $\neg A \wedge \neg B$: the number is not a multiple of 2 and not a multiple of 5.

(b) From 1 to 24 there are 12 multiples of 2 and 4 multiples of 5. The numbers that are multiples of both are the multiples of 10, which appear in both lists; counting them gives 2. So $12 + 4 - 2 = 14$ tickets make $A \vee B$ true. The other $24 - 14 = 10$ tickets make it false, so 10 tickets win.

SCORING

1 point: (a) the "and" of the two negations. 1 point: (b) 10 tickets.

4. Decide whether the two statements in each pair are contradictory. When a pair is not, name every row (letters P, Q in the fixed order) where the two statements have the same truth value. (3 points)

(a) $P \leftrightarrow Q$ and $(P \wedge \neg Q) \vee (\neg P \wedge Q)$

(b) $P \wedge Q$ and $\neg P \wedge \neg Q$

(c) $\neg P \rightarrow Q$ and $\neg P \wedge \neg Q$

ANSWER

(a) contradictory (b) not contradictory: the same value in rows 2 and 3 (c) contradictory

SOLUTION

Two statements are contradictory when they have opposite truth values in every row.

(a)

Row	P	Q	$P \leftrightarrow Q$	$(P \wedge \neg Q) \vee (\neg P \wedge Q)$
1	T	T	T	F
2	T	F	F	T
3	F	T	F	T
4	F	F	T	F

Opposite in every row: contradictory.

(b)

Row	P	Q	$P \wedge Q$	$\neg P \wedge \neg Q$
1	T	T	T	F
2	T	F	F	F
3	F	T	F	F
4	F	F	F	T

Both are F in rows 2 and 3, so they are not contradictory. ("Both true" and "both false" do not cover the rows where P and Q differ.)

(c)

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Row	P	Q	$\neg P \to Q$	$\neg P \wedge \neg Q$
1	T	T	T	F
2	T	F	T	F
3	F	T	T	F
4	F	F	F	T

Opposite in every row: contradictory. $\neg P \to Q$ fails only when P and Q are both F, which is the one row where $\neg P \wedge \neg Q$ holds.

SCORING

1 point for each part: (a) contradictory; (b) not contradictory, with rows 2 and 3; (c) contradictory.

5. For each statement, find every combination of truth values of its letters that gives the statement the value stated. (3 points)

(a) $\neg(P \vee \neg Q)$ is true.

(b) $\neg(P \wedge Q) \wedge P$ is true.

(c) $\neg(P \vee Q) \vee R$ is false. Name the rows of the table on P, Q, R (in the fixed order).

ANSWER

(a) P false, Q true (b) P true, Q false (c) R false and at least one of P, Q true: rows 2, 4 and 6

SOLUTION

(a) By the "or" law, $\neg(P \vee \neg Q) \equiv \neg P \wedge Q$. An "and" is true only when both parts are, so P is false and Q is true. No other combination works.

(b) An "and" is true only when both parts are, so P is true and $\neg(P \wedge Q)$ is true. By the "and" law, $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$; with $\neg P$ false, $\neg Q$ must be true, so Q is false.

(c) An "or" is false only when both parts are false, so R is false and $\neg(P \vee Q)$ is false. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ is false exactly when at least one of P, Q is true. So R is false and P, Q are not both false: P, Q, R = T, T, F (row 2), T, F, F (row 4) or F, T, F (row 6).

SCORING

1 point for each part: (a) P false, Q true; (b) P true, Q false; (c) rows 2, 4 and 6 and no others.

6. Priya keeps two rules for her garden: "If the soil is dry, I water the beds" and "If I water the beds, the level in the rain barrel falls." Let D be "the soil is dry", W be "Priya waters the beds" and B be "the level in the rain barrel falls". (3 points)

(a) One evening the level has not fallen, and Priya argues from her two rules: "So the soil was not dry." Write her argument as one conditional and decide, without writing all eight rows, whether it is valid.

(b) Her neighbour Omar argues from the same two rules: "The soil was not dry. So the level did not fall." Write his argument as one conditional and decide whether it is valid. If it is not, name every counterexample row (letters D, W, B in the fixed order).

ANSWER

(a) $\big((D \to W) \wedge (W \to B) \wedge \neg B\big) \to \neg D$; valid (b) $\big((D \to W) \wedge (W \to B) \wedge \neg D\big) \to \neg B$; invalid, counterexamples: rows 5 and 7

SOLUTION

(a) The conditional is $\big((D \to W) \wedge (W \to B) \wedge \neg B\big) \to \neg D$. It is false only on a row where every premise is T and the conclusion $\neg D$ is F. Then D is T, so $D \to W$ forces W T, and $W \to B$ forces B T. But $\neg B$ needs B F. No row can do both, so the conditional is a tautology and the argument is valid.

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(b) The conditional is $\big((D \to W) \wedge (W \to B) \wedge \neg D\big) \to \neg B$. A counterexample row has $\neg D$ T and $\neg B$ F, so D F and B T. Then $D \to W$ is T because D is F, and $W \to B$ is T because B is T, whatever W is. So both W T (row 5) and W F (row 7) make every premise true and the conclusion false. The argument is invalid: the level can fall for a reason other than dry soil, for example after Priya waters for no reason, or after someone else draws water.

SCORING
1 point: both conditionals written correctly. 1 point: (a) valid, with the clash on B found. 1 point: (b) invalid, rows 5 and 7 and no others.

7. For each statement below, say whether it is the converse, the inverse or the contrapositive of $P \to \neg Q$ (after cancelling any double negation), or none of these. Then say which of the four statements are logically equivalent to $P \to \neg Q$.

(a) $Q \to \neg P$ (b) $\neg Q \to P$ (c) $\neg P \to Q$ (d) $\neg P \to \neg Q$ (3 points)

ANSWER
(a) contrapositive (b) converse (c) inverse (d) none of these Of the four, only (a) is logically equivalent to $P \to \neg Q$.

SOLUTION
The hypothesis is P and the conclusion is $\neg Q$.

- Contrapositive, swap and negate both: $\neg\neg Q \to \neg P$, which is $Q \to \neg P$. That is (a).
- Converse, swap: $\neg Q \to P$. That is (b).
- Inverse, negate both: $\neg P \to \neg\neg Q$, which is $\neg P \to Q$. That is (c).
- (d) negates the hypothesis but leaves the conclusion $\neg Q$ as it was, so it is none of the three.

By the contrapositive law, (a) is logically equivalent to $P \to \neg Q$. Take P and Q both true: $P \to \neg Q$ is false, while (b), (c) and (d) are all true there, (b) because its conclusion P is true and (c) and (d) because their hypothesis $\neg P$ is false. So only (a) is equivalent to $P \to \neg Q$.

SCORING
1 point: (a) contrapositive and (b) converse. 1 point: (c) inverse and (d) none of these. 1 point: only (a) is equivalent, with a reason (the contrapositive law, or a row where the others differ).

8. A museum checks that this is true of every visitor: "If a visitor is under 12, then the visitor is with an adult." Which of the sentences below say the same thing as this rule? For each one that does not, describe a visitor who keeps the rule but breaks the sentence. (3 points)

(a) If a visitor is not with an adult, then the visitor is 12 or older.

(b) If a visitor is with an adult, then the visitor is under 12.

(c) If a visitor is 12 or older, then the visitor is not with an adult.

(d) No visitor under 12 is without an adult.

ANSWER
(a) and (d) say the same as the rule. (b) and (c) do not: a visitor aged 12 or older who is with an adult keeps the rule but breaks both.

SOLUTION
Let P be "the visitor is under 12" and Q be "the visitor is with an adult", so the rule is $P \to Q$. Its negations are $\neg P$, "the visitor is 12 or older", and $\neg Q$, "the visitor is not with an adult".

(a) This is $\neg Q \to \neg P$, the contrapositive: the same as the rule.

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(b) This is $Q \to P$, the converse. It is broken by a visitor with an adult who is not under 12, say a 15-year-old with a parent. That visitor keeps the rule, whose hypothesis is false for them.

(c) This is $\neg P \to \neg Q$, the inverse, logically equivalent to the converse. The same visitor breaks it: 12 or older, yet with an adult.

(d) The rule is false only for a visitor under 12 without an adult, and (d) says there is no such visitor: it rules out exactly the same case, so it says the same as the rule.

SCORING

1 point: (a) and (d) chosen as the same. 1 point: (b) and (c) rejected. 1 point: a visitor 12 or older who is with an adult (for both, or one for each).

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