Discrete Mathematics and Proof Unit 1 quiz 3: Saying a conditional without if-then, Exclusive or and logic circuits (answer key)

7 questions, 22 points

ANSWERS AT A GLANCE

1. (a) `=OR(NOT(B2), C2)` (b) `=AND(B2, NOT(C2))` (c) Every player without a signed medical form: for a player under 14 his formula says TRUE though the rule is broken, and for a player 14 or older it says FALSE though the rule is kept.
2. (a) $\neg P \wedge Q$ (b) $(P \vee Q) \wedge \neg R$ (c) $P \wedge Q \wedge \neg R$ (d) $(\neg P \vee Q) \wedge \neg R$
3. (a) $x^2 > 4x$ and $x \leq 4$; false, for example $x = -2$. (b) $x > 2$ and $x^2 \leq 2x$; none exists, the claim is true. (c) $x \neq 0$ and $x + \frac{1}{x} < 2$; false, for example $x = -1$.
4. (a) binary 11000, Gray code 10100 (b) $b_{i-1} \oplus g_i \equiv (b_{i-1} \oplus b_{i-1}) \oplus b_i \equiv b_i$; binary 10011, the number 19
5. (a) $M = (L \oplus R) \vee (O \wedge \neg B)$; 4 gates: XOR on L, R; NOT on B; AND on O and $\neg B$; OR on the two results (b) No: M is F (c) 7 gates
6. (a) lamp 1: $x_1 \oplus x_3$, lamp 2: $x_1 \oplus x_2$, lamp 3: $x_2 \oplus x_3$ (b) button 3 alone, or buttons 1 and 2 (c) the three lamps' statements have exclusive or F, so an even number of lamps is on.
7. See the proof below.

WORKED SOLUTIONS

1. A youth soccer league keeps a spreadsheet of its players. In each row, cell B is TRUE when the player is under 14, and cell C is TRUE when a signed medical form is on file. The league's rule is: "If a player is under 14, then a signed medical form is on file." The spreadsheet has the functions AND, OR and NOT, each giving TRUE or FALSE, but no if-then function. (3 points)

   (a) Write a formula using B2 and C2 that is TRUE exactly when the player in row 2 keeps the rule.

   (b) Write a formula using B2 and C2 that is TRUE exactly when the player in row 2 breaks the rule.

   (c) Kofi used `=OR(B2, C2)` for (a). Describe every kind of player for whom his formula gives the wrong answer.

   ANSWER

   (a) `=OR(NOT(B2), C2)` (b) `=AND(B2, NOT(C2))` (c) Every player without a signed medical form: for a player under 14 his formula says TRUE though the rule is broken, and for a player 14 or older it says FALSE though the rule is kept.

   SOLUTION

   Let P be "the player is under 14" (cell B) and Q be "a signed medical form is on file" (cell C). The rule is $P \to Q$.

   (a) $P \to Q \equiv \neg P \vee Q$, so the formula is `=OR(NOT(B2), C2)`.

   (b) A player breaks the rule when $\neg(P \to Q)$ is true, and $\neg(P \to Q) \equiv P \wedge \neg Q$: under 14 and no form. The formula is `=AND(B2, NOT(C2))`.

   (c) Kofi's formula is $P \vee Q$, which differs from $\neg P \vee Q$ only when Q is false, so only for players without a form. A player under 14 without a form breaks the rule, but $P \vee Q$ is TRUE. A player 14 or older without a form keeps the rule (its hypothesis is false), but $P \vee Q$ is FALSE. For every player with a form, both formulas are TRUE.

   SCORING

   1 point: (a) `=OR(NOT(B2), C2)` or an equivalent formula. 1 point: (b) `=AND(B2, NOT(C2))` or an equivalent formula. 1 point: (c) both kinds of player without a form, each with which way the formula is wrong.

2. Write the negation of each statement with no $\to$ in it and every $\neg$ on a single letter. (3 points)

   (a) $\neg P \to \neg Q$

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(b) $(P \vee Q) \to R$

(c) $P \to (Q \to R)$

(d) $(P \to Q) \to R$

ANSWER

(a) $\neg P \wedge Q$ (b) $(P \vee Q) \wedge \neg R$ (c) $P \wedge Q \wedge \neg R$ (d) $(\neg P \vee Q) \wedge \neg R$

SOLUTION

The negation of $A \to B$ is $A \wedge \neg B$: keep the hypothesis, negate the conclusion, and join them with and.

(a) $A = \neg P$, $B = \neg Q$: $\neg P \wedge \neg\neg Q$, which is $\neg P \wedge Q$.

(b) $A = P \vee Q$, $B = R$: $(P \vee Q) \wedge \neg R$.

(c) $A = P$, $B = Q \to R$: $P \wedge \neg(Q \to R)$, and the negation of $Q \to R$ is $Q \wedge \neg R$, so the answer is $P \wedge Q \wedge \neg R$.

(d) $A = P \to Q$, $B = R$: $(P \to Q) \wedge \neg R$. To remove the arrow, write $P \to Q$ in its or-form: $(\neg P \vee Q) \wedge \neg R$.

SCORING

1 point: (a) and (b). 1 point: (c), with the inner conditional negated as $Q \wedge \neg R$. 1 point: (d), with $P \to Q$ rewritten as $\neg P \vee Q$.

3. Each claim below is about every real number x. For each one, write the and-statement that a counterexample must satisfy. Then give a counterexample, or show that none exists, and say whether the claim is true. (3 points)

(a) If $x^2 > 4x$, then $x > 4$.

(b) If $x > 2$, then $x^2 > 2x$.

(c) If $x \neq 0$, then $x + \dfrac{1}{x} \geq 2$.

ANSWER

(a) $x^2 > 4x$ and $x \leq 4$; false, for example $x = -2$. (b) $x > 2$ and $x^2 \leq 2x$; none exists, the claim is true. (c) $x \neq 0$ and $x + \dfrac{1}{x} < 2$; false, for example $x = -1$.

SOLUTION

A counterexample to "if P then Q" is a value where the negation $P \wedge \neg Q$ holds.

(a) It must satisfy $x^2 > 4x$ and $x \leq 4$. At $x = -2$, $x^2 = 4 > -8 = 4x$ and $-2 \leq 4$. The claim is false (any $x < 0$ breaks it).

(b) It must satisfy $x > 2$ and $x^2 \leq 2x$. But when $x > 2$, both x and $x - 2$ are positive, so $x^2 - 2x = x(x - 2) > 0$, that is, $x^2 > 2x$. No value satisfies both halves, so there is no counterexample and the claim is true.

(c) It must satisfy $x \neq 0$ and $x + \dfrac{1}{x} < 2$. At $x = -1$, $x + \dfrac{1}{x} = -1 + (-1) = -2 < 2$. The claim is false (every negative x breaks it).

SCORING

1 point for each part: the and-statement, and a correct counterexample in (a) and (c) or the argument that none exists in (b), with the truth of the claim.

4. A rotary sensor on a robot arm reports each position as a 5-bit Gray code, so that neighbouring positions differ in exactly one bit. The binary number $b_1 b_2 b_3 b_4 b_5$ (b_1 on the left) has the Gray code $g_1 g_2 g_3 g_4 g_5$, where $g_1 = b_1$ and $g_i = b_{i-1} \oplus b_i$ for $i = 2, 3, 4, 5$. (3 points)

(a) Write 24 in 5-bit binary and find its Gray code.

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(b) Show that $b_i = b_{i-1} \oplus g_i$ for $i = 2, 3, 4, 5$. The sensor reports the Gray code 11010: find the binary number and its value.

ANSWER

(a) binary 11000, Gray code 10100 (b) $b_{i-1} \oplus g_i \equiv (b_{i-1} \oplus b_{i-1}) \oplus b_i \equiv b_i$; binary 10011, the number 19

SOLUTION

(a) 24 = 16 · 1 + 8 · 1 + 4 · 0 + 2 · 0 + 0, so the binary number is 11000. Then $g_1 = 1$, $g_2 = 1 \oplus 1 = 0$, $g_3 = 1 \oplus 0 = 1$, $g_4 = 0 \oplus 0 = 0$, $g_5 = 0 \oplus 0 = 0$. The Gray code is 10100: a 1 marks each place where neighbouring bits of the binary number differ.

(b) Exclusive-or both sides of $g_i = b_{i-1} \oplus b_i$ with b_{i-1}. By the associative law and $X \oplus X \equiv \mathrm{F}$, $b_{i-1} \oplus g_i \equiv (b_{i-1} \oplus b_{i-1}) \oplus b_i \equiv \mathrm{F} \oplus b_i \equiv b_i$.

So decode from the left: $b_1 = g_1 = 1$, $b_2 = 1 \oplus 1 = 0$, $b_3 = 0 \oplus 0 = 0$, $b_4 = 0 \oplus 1 = 1$, $b_5 = 1 \oplus 0 = 1$. The binary number is 10011, which is 16 · 1 + 8 · 0 + 4 · 0 + 2 · 1 + 1 = 19.

SCORING

1 point: (a) the binary number and its Gray code. 1 point: (b) $b_i = b_{i-1} \oplus g_i$ shown with the laws (associative law and $X \oplus X \equiv \mathrm{F}$). 1 point: (b) the decoded binary number and its value; one slip in the bits that is carried through correctly loses only this point.

5. A science-fair robot's motor has four inputs: L, "the left bump sensor is pressed"; R, "the right bump sensor is pressed"; O, "the remote override is on"; B, "the battery is low". The motor runs when exactly one bump sensor is pressed, or when the override is on and the battery is not low. (3 points)

(a) Write the motor's statement M, and describe a circuit for it with as few gates as you can (XOR gates allowed). How many gates?

(b) Both bump sensors are pressed, the override is on and the battery is low. Does the motor run?

(c) The parts bin has no XOR gates. Using the four-gate form of exclusive or, how many gates does the circuit need now?

ANSWER

(a) $M = (L \oplus R) \vee (O \wedge \neg B)$; 4 gates: XOR on L, R; NOT on B; AND on O and $\neg B$; OR on the two results (b) No: M is F (c) 7 gates

SOLUTION

(a) "Exactly one bump sensor is pressed" is $L \oplus R$. "The override is on and the battery is not low" is $O \wedge \neg B$. "Or" joins them: $M = (L \oplus R) \vee (O \wedge \neg B)$. The last connective is $\vee$, so the last gate is an OR. Its inputs come from an XOR gate fed by L and R, and an AND gate fed by O and by a NOT gate on B: 4 gates.

(b) $L \oplus R$ is $\mathrm{T} \oplus \mathrm{T}$, which is F. $O \wedge \neg B$ is $\mathrm{T} \wedge \mathrm{F}$, which is F. So M is $\mathrm{F} \vee \mathrm{F}$, F: the motor does not run.

(c) $L \oplus R \equiv (L \vee R) \wedge \neg(L \wedge R)$ takes an OR, an AND, a NOT and a second AND: 4 gates in place of 1. The rest of the circuit is unchanged, 3 gates, so 4 + 3 = 7.

SCORING

1 point: (a) M and a 4-gate circuit. 1 point: (b) F with both parts evaluated. 1 point: (c) 7, from the four-gate form of exclusive or.

6. A puzzle box has three lamps and three buttons, and all the lamps start off. Button 1 flips lamps 1 and 2, button 2 flips lamps 2 and 3, and button 3 flips lamps 1 and 3 (a flip turns a lamp that is off on, and one that is on off). Let x_1, x_2, x_3 be "button 1 (or 2, or 3) was pressed an odd number of times", and read a lamp as T when it is on. (3 points)

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(a) Write the state of each lamp as a statement in x_1, x_2, x_3, and use the laws of exclusive or to explain why the order of the presses does not matter.

(b) Find every set of buttons, each pressed once, that leaves lamps 1 and 3 on and lamp 2 off.

(c) Show that no sequence of presses leaves exactly one lamp on.

ANSWER

(a) lamp 1: $x_1 \oplus x_3$, lamp 2: $x_1 \oplus x_2$, lamp 3: $x_2 \oplus x_3$ (b) button 3 alone, or buttons 1 and 2 (c) the three lamps' statements have exclusive or F, so an even number of lamps is on.

SOLUTION

(a) Each press flips a lamp, that is, exclusive-ors its state with T. Lamp 1 starts at F and is flipped once for every press of button 1 or button 3, so its state is a chain $\mathrm{F} \oplus \mathrm{T} \oplus \cdots \oplus \mathrm{T}$. By the commutative and associative laws the T's can be put in any order, so the order of the presses does not matter, and since $\mathrm{T} \oplus \mathrm{T} \equiv \mathrm{F}$ the presses of one button cancel in pairs, leaving T exactly when that button was pressed an odd number of times. So lamp 1 is $x_1 \oplus x_3$. In the same way lamp 2 is $x_1 \oplus x_2$ and lamp 3 is $x_2 \oplus x_3$.

(b) Lamp 2 off means $x_1 \oplus x_2$ is F: x_1 and x_2 agree. Lamp 1 on means x_3 differs from x_1; then lamp 3, $x_2 \oplus x_3$, is on too, since x_2 agrees with x_1. So $x_1 = x_2$ and x_3 is the other value: either x_1, x_2 F and x_3 T (button 3 alone) or x_1, x_2 T and x_3 F (buttons 1 and 2). Check: buttons 1 and 2 flip lamp 1 once, lamp 2 twice and lamp 3 once.

(c) The exclusive or of the three lamps is $(x_1 \oplus x_3) \oplus (x_1 \oplus x_2) \oplus (x_2 \oplus x_3) \equiv (x_1 \oplus x_1) \oplus (x_2 \oplus x_2) \oplus (x_3 \oplus x_3) \equiv \mathrm{F}$. By the parity theorem, an even number of the lamps is on, 0 or 2, after any presses. Exactly one lamp on would be an odd number. (In words: every press flips two lamps, which never changes whether the number of lit lamps is even.)

SCORING

1 point: (a) the three statements, with the order explained by the commutative and associative laws. 1 point: (b) both sets of buttons. 1 point: (c) the exclusive or of the three lamps is F, so an even number are on, or the equivalent argument that each press flips two lamps.

7. You may use the law $(A \vee B) \wedge (A \vee C) \equiv A \vee (B \wedge C)$, the De Morgan laws, and the fact that an or with one part that is false in every row has the same truth value as its other part. (4 points)

(a) Without a truth table, show that $(P \to Q) \wedge (P \to \neg Q) \equiv \neg P$.

(b) Suppose that, starting from an assumption P, you have shown both that Q follows and that $\neg Q$ follows. Use (a) to say what you now know about P, and why.

ANSWER

(a) $(\neg P \vee Q) \wedge (\neg P \vee \neg Q) \equiv \neg P \vee (Q \wedge \neg Q) \equiv \neg P$, since $Q \wedge \neg Q$ is false in every row. (b) P is false: both conditionals are true, so by (a) $\neg P$ is true.

SOLUTION

(a) Write each conditional in its or-form, then use the given law with $A = \neg P$, $B = Q$ and $C = \neg Q$:

$$\begin{aligned}(P \to Q) \wedge (P \to \neg Q) &\equiv (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \\ &\equiv \neg P \vee (Q \wedge \neg Q)\end{aligned}$$

The part $Q \wedge \neg Q$ is false in every row, since Q and $\neg Q$ are never true together. An or with a part that is always false has the truth value of its other part, so the whole statement is equivalent to $\neg P$.

(b) Having shown that Q follows from P and that $\neg Q$ follows from P means $P \to Q$ and $P \to \neg Q$ are both true, so their conjunction is true. By (a) that conjunction is equivalent to $\neg P$, so $\neg P$ is true: P is false. An assumption that leads to a statement and to its negation must be false.

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SCORING

1 point: both conditionals rewritten as $\neg P \vee Q$ and $\neg P \vee \neg Q$. 1 point: the given law applied to reach $\neg P \vee (Q \wedge \neg Q)$. 1 point: $Q \wedge \neg Q$ is always false, so the statement is equivalent to $\neg P$. 1 point: (b) P is false, because the conjunction is true and it is equivalent to $\neg P$.

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