Discrete Mathematics and Proof Unit 1 test: Statements, truth tables and logical equivalence (answer key)

13 questions, 41 points

ANSWERS AT A GLANCE

1. (a) true (b) false
2. True when $-8 \le x < -4$ or $-2 < x \le 1$. False when $x < -8$ or $-4 \le x \le -2$ or $x > 1$.
3. (a) If n is not a multiple of 8, then n is not a multiple of 4 or not a multiple of 10. (b) $n = 20$ (c) No: the claim is false.
4. (a) $\neg P \to Q$ (b) $\neg(P \to \neg Q)$ (c) $\neg(\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P))$
5. (a) P false, Q false, R true (b) $P \to Q$ true, $R \to P$ false, $\neg P \to R$ true (c) 0, 1 and 2 (d) No
6. Last column F, F, T, F, T, T, T, F; true in rows 3, 5, 6 and 7.
7. (a) $H \wedge (U \to W)$ (b) last column T, F, T, T, F, F, F, F (c) Mason: row 2, may not climb. Leah: row 4, may climb.
8. (a) $W \to (R \to B)$ and $(W \wedge R) \to B$ (b) each is false only in row 2 (c) No: both are false in row 2 alone and true in every other row, so they have the same column.
9. Neither: false only in row 4.
10. (a) battery ≥ 20 percent and (wind ≤ 50 km/h or it is not raining) (b) battery ≥ 20 percent: true; wind ≤ 50 km/h: false; not raining: true; the drone keeps flying
11. (a) $(P \wedge \neg Q) \wedge (\neg P \vee Q)$: contradiction (b) $\neg P \vee \neg Q \vee P \vee Q$: tautology (c) $(\neg P \wedge \neg Q) \vee (\neg P \wedge Q)$: neither, true in rows 3 and 4
12. (a) 7 = 0111, 14 = 1110; $c_1 = 0$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$; $s_3 s_2 s_1 s_0 = 0101$ (b) $c_4 s_3 s_2 s_1 s_0 = 10101$, which is 21 = 7 + 14
13. (a) $L \leftrightarrow (C \wedge D)$ and $C \to (L \leftrightarrow D)$ (b) not equivalent: the biconditional is false in rows 5 and 7 (c) a student with no library card who is allowed to borrow a laptop, trained or not: the library's rule is broken, Jin's version is not.

WORKED SOLUTIONS

1. Let $n = 69$. Let P be "n is even", Q be "$n > 50$" and R be "n is a multiple of 3". Find the truth value of each statement, and show how you decided. (3 points)

(a) $(P \vee Q) \to R$

(b) $(P \wedge \neg R) \leftrightarrow Q$

ANSWER

(a) true (b) false

SOLUTION

First settle the three letters. 69 is odd, so P is false. 69 > 50, so Q is true. Dividing 69 by 3 leaves remainder 0, so R is true.

(a) R is true, and a conditional with a true conclusion is **true**.

(b) P is false, so $P \wedge \neg R$ is false, while Q is true. The two sides differ: **false**.

SCORING

1 point: the truth values of P, Q and R, each with its reason. 1 point: (a) true, from the value of $P \vee Q$ and the rule for a conditional. 1 point: (b) false, from the value of $P \wedge \neg R$ and the rule for a biconditional.

2. For which real numbers x is the sentence below true, and for which is it false? Describe each set with inequalities joined by "and" or "or".

$$(x < -4 \text{ or } x > -2) \text{ and } -8 \le x \le 1$$

(3 points)

ANSWER

True when $-8 \le x < -4$ or $-2 < x \le 1$. False when $x < -8$ or $-4 \le x \le -2$ or $x > 1$.

SOLUTION

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The sentence is an and of two parts: the bracket "$x < -4$ or $x > -2$", and "$-8 \leq x \leq 1$", which is itself the conjunction "$x \geq -8$ and $x \leq 1$".

The bracket is an or, so it fails only where both of its parts fail: where $x \geq -4$ and $x \leq -2$, that is, for $-4 \leq x \leq -2$. Everywhere else it is true.

An and is true only where both parts are true, so the sentence is true for the numbers from −8 to 1 that are not from −4 to −2. Since −8 < −4 < −2 < 1, that leaves two pieces:

$$-8 \leq x < -4 \quad \text{or} \quad -2 < x \leq 1$$

The endpoints −4 and −2 are left out, since there both parts of the bracket fail; −8 and 1 stay in because the second part uses $\leq$.

An and is false where at least one part is false. The second part fails for $x < -8$ and for $x > 1$; the bracket fails for $-4 \leq x \leq -2$. So the sentence is false when $x < -8$ or $-4 \leq x \leq -2$ or $x > 1$, which is every real number not in the true set.

SCORING

1 point: the true set, $-8 \leq x < -4$ or $-2 < x \leq 1$, with each endpoint correctly kept or left out. 1 point: the false set, all three pieces. 1 point: reasoning from the truth conditions: the bracket fails only where both of its parts fail, and the and needs both of its parts.

3. For positive integers n, consider the claim: "if n is a multiple of 4 and a multiple of 10, then n is a multiple of 8." (3 points)

(a) Write its contrapositive as an if-then sentence. Do not use the words "not both" or "it is not the case".

(b) Find the smallest positive integer n that makes the contrapositive false, or show that no such n exists.

(c) Is the claim itself true? Answer from (b), without testing the claim again.

ANSWER

(a) If n is not a multiple of 8, then n is not a multiple of 4 or not a multiple of 10. (b) $n = 20$ (c) No: the claim is false.

SOLUTION

(a) The conclusion negated is "n is not a multiple of 8". The hypothesis is an and, so its negation is an or (the De Morgan laws): "n is not a multiple of 4 or not a multiple of 10". The contrapositive: if n is not a multiple of 8, then n is not a multiple of 4 or not a multiple of 10.

(b) The contrapositive is false only when its hypothesis is true and its conclusion is false: n is not a multiple of 8, and n is a multiple of both 4 and 10. The positive integers that are multiples of both 4 and 10 are the multiples of their least common multiple, 20. 20 is not a multiple of 8, so $n = 20$ makes the contrapositive false, and no smaller positive integer is a multiple of both 4 and 10.

(c) A conditional and its contrapositive are true together and false together, so the claim is false too, broken by the same $n = 20$: a multiple of 4 and of 10 that is not a multiple of 8.

SCORING

1 point: (a) the contrapositive, with the and in the hypothesis turned into an or. 1 point: (b) $n = 20$, with the reason. 1 point: (c) No: the claim is false. A conditional and its contrapositive always have the same truth value.

4. (3 points)

(a) Write $P \vee Q$ using only the connectives $\neg$ and $\rightarrow$.

(b) Write $P \wedge Q$ using only the connectives $\neg$ and $\rightarrow$.

(c) Write $P \leftrightarrow Q$ using only the connectives $\neg$ and $\vee$.

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Show briefly why each of your answers is equivalent to the statement it replaces.

ANSWER

(a) $\neg P \to Q$ (b) $\neg(P \to \neg Q)$ (c) $\neg(\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P))$

SOLUTION

(a) The or-form of $\neg P \to Q$ is $\neg\neg P \vee Q$, which is $P \vee Q$. So $P \vee Q \equiv \neg P \to Q$.

(b) The negation of a conditional is a conjunction: $\neg(A \to B) \equiv A \wedge \neg B$. With $A = P$ and $B = \neg Q$ this gives $\neg(P \to \neg Q) \equiv P \wedge \neg\neg Q \equiv P \wedge Q$.

(c) A biconditional says both conditionals: $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$. To remove the and, use the De Morgan law in the form $A \wedge B \equiv \neg(\neg A \vee \neg B)$, with $A = \neg P \vee Q$ and $B = \neg Q \vee P$:

$$P \leftrightarrow Q \;\equiv\; \neg(\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P))$$

SCORING

1 point each for (a), (b) and (c), with a correct reason; any equivalent answer that uses only the allowed connectives earns the point.

5. Let $x = -1$, and let P be "$x^2 > 4$", Q be "$x > 2$" and R be "$x < 0$". (4 points)

(a) Give the truth values of P, Q and R.

(b) Give the truth values of $P \to Q$, $R \to P$ and $\neg P \to R$.

(c) Now let x be any integer, with P and R the same sentences about it. Find every integer x for which $\neg P \to R$ is false.

(d) Is there any integer x for which $Q \to P$ is false? Explain.

ANSWER

(a) P false, Q false, R true (b) $P \to Q$ true, $R \to P$ false, $\neg P \to R$ true (c) 0, 1 and 2 (d) No

SOLUTION

(a) $x^2 = 1$, so "$x^2 > 4$" is false. "−1 > 2" is false, and "−1 < 0" is true.

(b) A conditional is false only when its hypothesis is true and its conclusion is false. $P \to Q$ has hypothesis false and conclusion false: true. $R \to P$ has hypothesis true and conclusion false: false. $\neg P \to R$ has hypothesis $\neg P$ true and conclusion true: true.

(c) $\neg P \to R$ is false exactly when $\neg P$ is true and R is false, that is, $x^2 \leq 4$ and $x \geq 0$. For $x \geq 0$, $x^2 \leq 4$ means $x \leq 2$. So it is false for x = 0, 1 and 2, and for no other integer.

(d) No. $Q \to P$ would be false only with $x > 2$ and $x^2 \leq 4$. But if $x > 2$, then x is positive and $x \cdot x > 2 \cdot x > 2 \cdot 2 = 4$, so P is true. Every integer lands in a true row of $Q \to P$.

SCORING

1 point each: (a) P false, Q false, R true; (b) all three values right; (c) 0, 1 and 2 and no others; (d) no, because $x > 2$ forces $x^2 > 4$.

6. Build a truth table for $(P \to \neg Q) \leftrightarrow (Q \vee R)$, with a column for every piece, rows in the fixed order P, Q, R. Then list the rows in which the statement is true. (3 points)

ANSWER

Last column F, F, T, F, T, T, T, F; true in rows 3, 5, 6 and 7.

SOLUTION

$\neg Q$ comes first, then $P \to \neg Q$, false only where P is true and $\neg Q$ is false. $Q \vee R$ needs only the letters. The biconditional is true where those two columns agree.

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Row	P	Q	R	$\neg Q$	$P \to \neg Q$	$Q \vee R$	$(P \to \neg Q) \leftrightarrow (Q \vee R)$
1	T	T	T	F	F	T	F
2	T	T	F	F	F	T	F
3	T	F	T	T	T	T	T
4	T	F	F	T	T	F	F
5	F	T	T	F	T	T	T
6	F	T	F	F	T	T	T
7	F	F	T	T	T	T	T
8	F	F	F	T	T	F	F

The statement is true in rows 3, 5, 6 and 7.

SCORING

1 point: eight rows in the fixed order and a correct $\neg Q$ and $Q \vee R$. 1 point: correct $P \to \neg Q$ (false only in rows 1 and 2). 1 point: correct last column and the true rows 3, 5, 6, 7 (follow-through from one earlier slip loses only that point).

7. A climbing gym's rule: a climber may climb when the climber is wearing a harness and, if the climber is under 16, a parent has signed the waiver. Let H be "the climber is wearing a harness", U be "the climber is under 16" and W be "a parent has signed the waiver". (3 points)

(a) Write the rule as a statement in H, U and W.

(b) Build its truth table, rows in the order H, U, W.

(c) Mason is 15, is wearing a harness and has no signed waiver. Leah is 17, is wearing a harness and has no signed waiver. Name each one's row and say whether each may climb.

ANSWER

(a) $H \wedge (U \to W)$ (b) last column T, F, T, T, F, F, F, F (c) Mason: row 2, may not climb. Leah: row 4, may climb.

SOLUTION

(a) The harness is always needed, and the waiver is needed only under the condition "under 16", which is a conditional: $H \wedge (U \to W)$.

(b) $U \to W$ is false only where U is true and W is false; the last column is the and of H with it.

Row	H	U	W	$U \to W$	$H \wedge (U \to W)$
1	T	T	T	T	T
2	T	T	F	F	F
3	T	F	T	T	T
4	T	F	F	T	T
5	F	T	T	T	F
6	F	T	F	F	F
7	F	F	T	T	F
8	F	F	F	T	F

(c) Mason has H true, U true, W false: row 2, where the rule is F, so he may not climb. Leah has H true, U false, W false: row 4. There the conditional $U \to W$ is true because its hypothesis is false, so the rule is T and she may climb; the waiver does not apply to her.

SCORING

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1 point: (a) $H \wedge (U \to W)$ or an equivalent statement. 1 point: (b) a correct table with last column T, F, T, T, F, F, F, F. 1 point: (c) Mason row 2, may not climb; Leah row 4, may climb.

8. Diego's rule: "If it is a weekday, then if it is raining, I take the bus." His sister Leah writes it as: "If it is a weekday and it is raining, I take the bus." Let W be "it is a weekday", R be "it is raining" and B be "Diego takes the bus". (3 points)

(a) Write each version as a statement in W, R and B.

(b) Without writing the whole table, find every row (letters in the order W, R, B) in which each statement is false.

(c) Do the two versions ever say different things? Explain.

ANSWER

(a) $W \to (R \to B)$ and $(W \wedge R) \to B$ (b) each is false only in row 2 (c) No: both are false in row 2 alone and true in every other row, so they have the same column.

SOLUTION

(a) Diego's version has a conditional as its conclusion: $W \to (R \to B)$. Leah's has an and as its hypothesis: $(W \wedge R) \to B$.

(b) $W \to (R \to B)$ is false only where W is true and $R \to B$ is false, and $R \to B$ is false only where R is true and B is false. So W true, R true, B false: row 2, and no other.

$(W \wedge R) \to B$ is false only where $W \wedge R$ is true, so W and R are both true, and B is false: again row 2 alone.

(c) No. Each column is F in row 2 and T in the other seven rows, so the two statements have the same truth value in every row.

SCORING

1 point: (a) both statements. 1 point: (b) row 2 for each, found from hypothesis true and conclusion false. 1 point: (c) no, same column.

9. Decide whether $\big(P \wedge (Q \to R)\big) \to (P \wedge R)$ is a tautology, a contradiction or neither (letters P, Q, R in the fixed order). If it is neither, name every row where it is false. You need not write all eight rows, but your reasoning must account for every row. (3 points)

ANSWER

Neither: false only in row 4.

SOLUTION

Try to make it false. The hypothesis needs P true and $Q \to R$ true; the conclusion $P \wedge R$ false then needs R false, and $Q \to R$ with R false forces Q false. Every letter is forced, so only row 4 is false, and row 1, for one, is true. Neither: false only in row 4.

SCORING

1 point: reasoning that accounts for every row (a hunt for a false or a true row that follows every forced letter, or a full table). 1 point: the right kind. 1 point: the false rows named exactly, or the clash that rules out every row, as in: Neither: false only in row 4.

10. A delivery drone must land when its battery is below 20 percent, or when the wind speed is above 50 km/h while it is raining. Otherwise it keeps flying. (3 points)

(a) Use the De Morgan laws to write the condition for the drone to keep flying, with no "not" in front of a bracketed condition and each inequality written without "not".

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(b) The drone reads a battery level of 25 percent and a wind speed of 55 km/h, and it is not raining. Give the truth value of each simple part of your condition in (a), and say whether the drone keeps flying.

ANSWER

(a) battery ≥ 20 percent and (wind ≤ 50 km/h or it is not raining) (b) battery ≥ 20 percent: true; wind ≤ 50 km/h: false; not raining: true; the drone keeps flying

SOLUTION

(a) The landing condition is "battery below 20, or (wind above 50 and raining)". Keeping flying is its negation. The "or" law on the outside gives: not (battery below 20) and not (wind above 50 and raining). The first part is battery ≥ 20 percent. The "and" law on the second part gives: wind not above 50, that is wind ≤ 50 km/h, or it is not raining. So the drone keeps flying when battery ≥ 20 percent and (wind ≤ 50 km/h or it is not raining).

(b) 25 ≥ 20 is true; 55 ≤ 50 is false; "it is not raining" is true. The bracketed "or" needs one true part, and the outer "and" needs both of its parts, so the drone keeps flying.

SCORING

1 point: (a) both laws applied: the outer "or" becomes "and", the inner "and" becomes "or", and each inequality flips with the right strictness. 1 point: (b) the three truth values true, false, true. 1 point: the drone keeps flying.

11. Use the De Morgan laws to rewrite each statement with every $\neg$ on a single letter. Then, without a full truth table, classify it as a tautology, a contradiction or neither. If it is neither, name the rows (letters P, Q in the fixed order) where it is true. (3 points)

(a) $\neg(\neg P \vee Q) \wedge \neg(P \wedge \neg Q)$

(b) $\neg(P \wedge Q) \vee \neg(\neg P \wedge \neg Q)$

(c) $\neg(P \vee Q) \vee \neg(P \vee \neg Q)$

ANSWER

(a) $(P \wedge \neg Q) \wedge (\neg P \vee Q)$: contradiction (b) $\neg P \vee \neg Q \vee P \vee Q$: tautology (c) $(\neg P \wedge \neg Q) \vee (\neg P \wedge Q)$: neither, true in rows 3 and 4

SOLUTION

(a) The "or" law on the first part gives $\neg\neg P \wedge \neg Q$, which is $P \wedge \neg Q$; the "and" law on the second gives $\neg P \vee \neg\neg Q$, which is $\neg P \vee Q$. So the statement is $(P \wedge \neg Q) \wedge (\neg P \vee Q)$. The second part is exactly the negation of the first (that is what the "and" law just showed), so the statement has the form $X \wedge \neg X$: on every row one of the two parts is false. A contradiction.

(b) The "and" law on each part gives $(\neg P \vee \neg Q) \vee (\neg\neg P \vee \neg\neg Q)$, which is $\neg P \vee \neg Q \vee P \vee Q$. On every row one of P, $\neg P$ is true, and one true part makes an "or" true. A tautology.

(c) The "or" law on each part gives $(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg\neg Q)$, which is $(\neg P \wedge \neg Q) \vee (\neg P \wedge Q)$. Both parts need $\neg P$, so the statement is false whenever P is true (rows 1 and 2). When P is false, Q is either false or true, so one of the two parts holds. The statement is true exactly in rows 3 and 4: it says the same as $\neg P$. Neither.

SCORING

1 point for each part: the correct rewritten form and its class; for (c), rows 3 and 4 and no others.

12. A 4-bit adder adds $a = a_3a_2a_1a_0$ and $b = b_3b_2b_1b_0$ one column at a time, from the right, as you add by hand. Column i takes a_i, b_i and the carry c_i from the column to its right ($c_0 = 0$). It outputs the sum bit $s_i = a_i \oplus b_i \oplus c_i$ and passes on the carry c_{i+1}, which is 1 when at least two of a_i, b_i, c_i are 1. (3 points)

(a) Add 7 and 14 this way. Write both in 4-bit binary, and give c_1, c_2, c_3, c_4 and $s_3s_2s_1s_0$.

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(b) Check that the 5-bit number $c_4 s_3 s_2 s_1 s_0$ is 7 + 14.

ANSWER

(a) 7 = 0111, 14 = 1110; $c_1 = 0$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$; $s_3 s_2 s_1 s_0 = 0101$ (b) $c_4 s_3 s_2 s_1 s_0 = 10101$, which is 21 = 7 + 14

SOLUTION

(a) In binary 7 = 0111 and 14 = 1110. By the parity theorem the sum bit of a column is 1 when an odd number of its three bits are 1; the carry is 1 when at least two are. Work from the right, with $c_0 = 0$.

Column 0: $s_0 = 1 \oplus 0 \oplus 0 = 1$, and $c_1 = 0$.

Column 1: $s_1 = 1 \oplus 1 \oplus 0 = 0$, and $c_2 = 1$.

Column 2: $s_2 = 1 \oplus 1 \oplus 1 = 1$, and $c_3 = 1$.

Column 3: $s_3 = 0 \oplus 1 \oplus 1 = 0$, and $c_4 = 1$.

So $s_3 s_2 s_1 s_0 = 0101$.

(b) $c_4 s_3 s_2 s_1 s_0 = 10101$ is $16 \cdot 1 + 8 \cdot 0 + 4 \cdot 1 + 2 \cdot 0 + 1 = 21$, and 7 + 14 = 21.

SCORING

1 point: both numbers in 4-bit binary. 1 point: the four carries. 1 point: the sum bits and the check that $c_4 s_3 s_2 s_1 s_0$ is the sum; a slip carried through correctly loses only one point.

13. A library's rule: "A student may borrow a laptop if and only if the student has a library card and has done the laptop training." Jin rewrites it as: "If a student has a library card, then the student may borrow a laptop if and only if the student has done the training." Let C be "the student has a library card", D be "the student has done the training" and L be "the student may borrow a laptop". (4 points)

(a) Write the library's rule and Jin's version in symbols.

(b) Use a truth table, letters in the order C, D, L, to test whether the two versions are logically equivalent. Name every row where they disagree.

(c) Describe in words a student about whom the two versions disagree.

ANSWER

(a) $L \leftrightarrow (C \wedge D)$ and $C \rightarrow (L \leftrightarrow D)$ (b) not equivalent: the biconditional is false in rows 5 and 7 (c) a student with no library card who is allowed to borrow a laptop, trained or not: the library's rule is broken, Jin's version is not.

SOLUTION

(a) The library's rule is the biconditional $L \leftrightarrow (C \wedge D)$. Jin's version is the conditional $C \rightarrow (L \leftrightarrow D)$.

(b) The two are equivalent exactly when the biconditional joining them is a tautology.

Row	C	D	L	$C \wedge D$	$L \leftrightarrow (C \wedge D)$	$L \leftrightarrow D$	$C \rightarrow (L \leftrightarrow D)$	$\big(L \leftrightarrow (C \wedge D)\big) \leftrightarrow \big(C \rightarrow (L \leftrightarrow D)\big)$
1	T	T	T	T	T	T	T	T
2	T	T	F	T	F	F	F	T
3	T	F	T	F	F	F	F	T
4	T	F	F	F	T	T	T	T
5	F	T	T	F	F	T	T	F
6	F	T	F	F	T	F	T	T
7	F	F	T	F	F	F	T	F
8	F	F	F	F	T	T	T	T

The biconditional is F in rows 5 and 7, so the versions are not logically equivalent. In rows 1 to 4, where C is T, both versions come down to $L \leftrightarrow D$ and agree.

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(c) Rows 5 and 7 have C F and L T: a student with no library card who is allowed to borrow a laptop, whether or not the student has done the training. The library's rule says this must not happen. Jin's version has a false hypothesis there, so it is true and says nothing about students without a card.

SCORING

1 point: (a) both statements. 1 point: (b) correct columns for both versions. 1 point: (b) not equivalent, rows 5 and 7 and no others. 1 point: (c) no card but allowed to borrow, with which version it breaks.

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