Discrete Mathematics and Proof Unit 1 worksheet: Building a truth table for a compound statement (answer key)

ANSWERS AT A GLANCE

1. (a) last column F, F, F, T (b) last column T, F, T, F, F, F, T, T
2. (a) 8 rows, 7 columns (b) 8 rows, 8 columns (c) 16 rows, 9 columns
3. Row 3: the entry for $\neg P \vee Q$ should be F. Row 6: the entry for $Q \wedge R$ should be F. Correct last column: F, T, F, F, F, T, T, T.
4. (a) $(D \wedge A) \vee B$ and $D \wedge (A \vee B)$ (b) first reading T, T, T, F, T, F, T; second reading T, T, T, F, F, F, F, F (c) rows 5 and 7: the back door is closed and the panic button is pressed. The first reading sounds the alarm there and the second does not. The shop most likely means the first.
5. (a) In row 1 every column starts from T's, and T and T give T under both $\wedge$ and $\vee$; in row 4 F and F give F under both. (b) $\neg P$ is F in row 1, and $P \wedge \neg Q$ is F in row 1, so neither column can be the column of such a statement.

WORKED SOLUTIONS

1. Build a truth table for each statement, with a column for every piece. A piece that appears twice gets its column only once.

   (a) $\neg P \wedge \neg(Q \wedge \neg P)$

   (b) $(P \vee \neg Q) \wedge (\neg P \vee R)$

   ANSWER

   (a) last column F, F, F, T (b) last column T, F, T, F, F, F, T, T

   SOLUTION

   (a) Two letters, $2^2 = 4$ rows. $\neg P$ is built once and read twice: inside $Q \wedge \neg P$, and again at the end. Then $Q \wedge \neg P$ is negated, and the last column joins $\neg P$ with that negation by and.

Row	P	Q	$\neg P$	$Q \wedge \neg P$	$\neg(Q \wedge \neg P)$	$\neg P \wedge \neg(Q \wedge \neg P)$
1	T	T	F	F	T	F
2	T	F	F	F	T	F
3	F	T	T	T	F	F
4	F	F	T	F	T	T

   (b) Three letters, $2^3 = 8$ rows. $\neg Q$ and $\neg P$ come straight from the letters; $P \vee \neg Q$ and $\neg P \vee R$ each need one of them; the last column is the and of those two.

Row	P	Q	R	$\neg Q$	$\neg P$	$P \vee \neg Q$	$\neg P \vee R$	$(P \vee \neg Q) \wedge (\neg P \vee R)$
1	T	T	T	F	F	T	T	T
2	T	T	F	F	F	T	F	F
3	T	F	T	T	F	T	T	T
4	T	F	F	T	F	T	F	F
5	F	T	T	F	T	F	T	F
6	F	T	F	F	T	F	T	F
7	F	F	T	T	T	T	T	T
8	F	F	F	T	T	T	T	T

2. Do not fill in these tables. For each statement, say how many rows its truth table needs, list every column after the letters in an order in which they can be filled, building a repeated piece only once, and say how many columns the table has in all, letters included (do not count the row-number column).

   (a) $(P \wedge \neg Q) \vee (\neg Q \wedge R)$

   (b) $\neg\big(Q \vee \neg(P \wedge R)\big) \vee (P \wedge R)$

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(c) $(P \vee \neg S) \wedge \neg(Q \vee R)$

ANSWER

(a) 8 rows, 7 columns (b) 8 rows, 8 columns (c) 16 rows, 9 columns

SOLUTION

(a) Three letters, so $2^3 = 8$ rows. Columns after the letters: $\neg Q$; $P \wedge \neg Q$; $\neg Q \wedge R$; the whole statement. $\neg Q$ appears twice but is built once, and it must come first, since both conjunctions need it. 3 + 4 = 7 columns.

(b) Three letters, 8 rows. Columns: $P \wedge R$; $\neg(P \wedge R)$; $Q \vee \neg(P \wedge R)$; $\neg\big(Q \vee \neg(P \wedge R)\big)$; the whole statement. $P \wedge R$ is built once and read twice, inside the negation and again at the end. 3 + 5 = 8 columns.

(c) Four letters, $2^4 = 16$ rows. Columns: $\neg S$; $P \vee \neg S$; $Q \vee R$; $\neg(Q \vee R)$; the whole statement. The two halves do not depend on each other, so either may be built first. 4 + 5 = 9 columns.

3. Kofi built this table for $(\neg P \vee Q) \wedge \neg(Q \wedge R)$. He made exactly two mistakes, each in a single entry, and filled every later column correctly from what he had already written.

Row	P	Q	R	$\neg P$	$\neg P \vee Q$	$Q \wedge R$	$\neg(Q \wedge R)$	$(\neg P \vee Q) \wedge \neg(Q \wedge R)$
1	T	T	T	F	T	T	F	F
2	T	T	F	F	T	F	T	T
3	T	F	T	F	T	F	T	T
4	T	F	F	F	F	F	T	F
5	F	T	T	T	T	T	F	F
6	F	T	F	T	T	T	F	F
7	F	F	T	T	T	F	T	T
8	F	F	F	T	T	F	T	T

Find each mistake: give its row, the column it is in and the correct entry. Then write the correct last column.

ANSWER

Row 3: the entry for $\neg P \vee Q$ should be F. Row 6: the entry for $Q \wedge R$ should be F. Correct last column: F, T, F, F, F, T, T, T.

SOLUTION

Check each column against the columns it is built from, row by row.

Row 3: $\neg P$ is F and Q is F, so $\neg P \vee Q$ is F, not T. Kofi seems to have read P in place of $\neg P$. His last entry, T, followed from the wrong T; with F there, the last entry is F.

Row 6: Q is T and R is F, so $Q \wedge R$ is F, not T (that would be the value of $Q \vee R$). Then $\neg(Q \wedge R)$ is T, and the last entry is T $\wedge$ T, which is T.

Every other entry agrees with the columns it is built from. The corrected table:

Row	P	Q	R	$\neg P$	$\neg P \vee Q$	$Q \wedge R$	$\neg(Q \wedge R)$	$(\neg P \vee Q) \wedge \neg(Q \wedge R)$
1	T	T	T	F	T	T	F	F
2	T	T	F	F	T	F	T	T
3	T	F	T	F	F	F	T	F
4	T	F	F	F	F	F	T	F
5	F	T	T	T	T	T	F	F
6	F	T	F	T	T	F	T	T
7	F	F	T	T	T	F	T	T
8	F	F	F	T	T	F	T	T

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4. A shop's instructions say: "The alarm sounds when the back door is opened and the system is armed or the panic button is pressed." Let D be "the back door is opened", A be "the system is armed" and B be "the panic button is pressed".

(a) The sentence can be grouped in two ways. Write both as statements with parentheses.

(b) Build one truth table, in the order D, A, B, with a last column for each reading.

(c) In which rows do the two readings differ? Describe those situations in words, and say which reading the shop most likely means.

ANSWER

(a) $(D \wedge A) \vee B$ and $D \wedge (A \vee B)$ (b) first reading T, T, T, F, T, F, T, F; second reading T, T, T, F, F, F, F, F (c) rows 5 and 7: the back door is closed and the panic button is pressed. The first reading sounds the alarm there and the second does not. The shop most likely means the first.

SOLUTION

(a) Either "door and armed" is one piece and the panic button is the other choice, $(D \wedge A) \vee B$, or the door is required and "armed or panic" is the second piece, $D \wedge (A \vee B)$.

(b)

Row	D	A	B	$D \wedge A$	$(D \wedge A) \vee B$	$A \vee B$	$D \wedge (A \vee B)$
1	T	T	T	T	T	T	T
2	T	T	F	T	T	T	T
3	T	F	T	F	T	T	T
4	T	F	F	F	F	F	F
5	F	T	T	F	T	T	F
6	F	T	F	F	F	T	F
7	F	F	T	F	T	T	F
8	F	F	F	F	F	F	F

(c) The two last columns differ only in rows 5 and 7. In both, D is false and B is true: the back door is closed and someone presses the panic button (with the system armed in row 5, not armed in row 7). The first reading sounds the alarm; the second does not, because it needs the door open. A panic button that only works while the back door is open would be of little use, so the shop most likely means $(D \wedge A) \vee B$.

5. A statement is built from the letters P and Q using only $\wedge$ and $\vee$, as many times as you like and with any parentheses, but never $\neg$.

(a) Explain why its last column must be T in row 1, where both letters are true, and F in row 4, where both are false.

(b) Use (a) to explain why no statement built this way has the same column as $\neg P$, and none has the same column as $P \wedge \neg Q$.

ANSWER

(a) In row 1 every column starts from T's, and T and T give T under both $\wedge$ and $\vee$; in row 4 F and F give F under both. (b) $\neg P$ is F in row 1, and $P \wedge \neg Q$ is F in row 1, so neither column can be the column of such a statement.

SOLUTION

(a) Build the table column by column. In row 1 the two letter columns both hold T. Every later column is filled from two columns already on the page, by the table for $\wedge$ or for $\vee$. If both of those hold T in row 1, the new column does too, since T $\wedge$ T and T $\vee$ T are both T. So, column after column, row 1 stays T, up to

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the last column. In row 4 the letters both hold F, and F $\wedge$ F and F $\vee$ F are both F, so the same argument keeps every column F there.

(b) The column of $\neg P$ is F, F, T, T. It has F in row 1 (and T in row 4), so by (a) it is not the column of any statement built from P and Q with only $\wedge$ and $\vee$. The column of $P \wedge \neg Q$ is F, T, F, F: it has F in row 1 as well, so it is ruled out the same way. Some $\neg$ is needed to write either one.

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