Discrete Mathematics and Proof Unit 1 worksheet: Saying a conditional without if-then (answer key)

ANSWERS AT A GLANCE

1. (a) $P \vee Q$ (b) $\neg P \vee \neg Q$ (c) $R \to S$ (d) $\neg P \to \neg Q$, or equally $Q \to P$ (e) $\neg P \vee \neg Q \vee R$
2. (a) 91 is odd and 91 is not prime. The negation is true ($91 = 7 \times 13$). (b) $2^{10} > 1000$ and $2^{20} \leq 1{,}000{,}000$. The statement is true. (c) $\sqrt{50} < 7$ and $50 \geq 49$. The statement is true. (d) The pool is heated and the swim meet does not go ahead. (e) Naomi does not practice and she passes her driving test.
3. (a) If the bakery does not sell out by noon, then Hannah buys bread there after school. (b) If the bridge is open, then the ferry does not run. (c) If Elena gets a ticket, then she goes to the concert. (d) If n is even, then n^2 is a multiple of 4; true for every integer. (e) If n is not a multiple of 3, then $n^2 + 1$ is even; false, for example $n = 2$.
4. (a) $P \to \neg P \equiv \neg P \vee \neg P \equiv \neg P$: a statement that implies its own negation is false. (b) Both sides are $\neg P \vee \neg Q \vee R$. (c) Both sides are $(\neg P \wedge \neg Q) \vee R$.
5. See the proof below.

WORKED SOLUTIONS

1. Rewrite each conditional as an or-statement, and each or-statement as a conditional. Keep every ¬ on a single letter, with no letter carrying two of them.

   (a) $\neg P \to Q$

   (b) $P \to \neg Q$

   (c) $\neg R \vee S$

   (d) $P \vee \neg Q$

   (e) $(P \wedge Q) \to R$

   ANSWER

   (a) $P \vee Q$ (b) $\neg P \vee \neg Q$ (c) $R \to S$ (d) $\neg P \to \neg Q$, or equally $Q \to P$ (e) $\neg P \vee \neg Q \vee R$

   SOLUTION

   A conditional $A \to B$ is the same as $\neg A \vee B$. Read backward, an or-statement $A \vee B$ is the or-form of $\neg A \to B$, since $\neg\neg A$ is A.

   (a) $\neg P \to Q \equiv \neg\neg P \vee Q$, and the double negation cancels: $P \vee Q$.

   (b) $P \to \neg Q \equiv \neg P \vee \neg Q$.

   (c) $\neg R \vee S$ has the shape $\neg A \vee B$ with $A = R$ and $B = S$, so it is $R \to S$.

   (d) Take $A = P$ and $B = \neg Q$: $P \vee \neg Q \equiv \neg P \to \neg Q$. The parts of an or may be listed in either order, so $P \vee \neg Q$ is also $\neg Q \vee P$, which is $Q \to P$: the contrapositive of the first answer, as it must be.

   (e) $(P \wedge Q) \to R \equiv \neg(P \wedge Q) \vee R$, and by the De Morgan law for and, $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$, so the statement is $\neg P \vee \neg Q \vee R$.

2. Write the negation of each statement as an and-sentence. Do not use "if" or "it is not the case". In (a) to (c), also decide which is true, the statement or its negation.

   (a) If 91 is odd, then 91 is prime.

   (b) If $2^{10} > 1000$, then $2^{20} > 1{,}000{,}000$.

   (c) If $\sqrt{50} < 7$, then 50 < 49.

   (d) If the pool is heated, then the swim meet goes ahead.

   (e) If Naomi does not practice, then she does not pass her driving test.

   ANSWER

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(a) 91 is odd and 91 is not prime. The negation is true (91 = 7 × 13). (b) $2^{10} > 1000$ and $2^{20} \leq 1{,}000{,}000$. The statement is true. (c) $\sqrt{50} < 7$ and $50 \geq 49$. The statement is true. (d) The pool is heated and the swim meet does not go ahead. (e) Naomi does not practice and she passes her driving test.

SOLUTION

The negation of "if P then Q" is "P and not Q": the hypothesis stays as it is, and the conclusion is negated.

(a) 91 is odd and 91 is not prime. Since 91 = 7 × 13, both halves hold, so the negation is true and the statement is false.

(b) $2^{10} > 1000$ and $2^{20} \leq 1{,}000{,}000$. Here $2^{10} = 1024$, so the hypothesis is true, and $2^{20} = 1024^2 = 1{,}048{,}576 > 1{,}000{,}000$, so the conclusion is true too. The statement is true and its negation is false.

(c) The negation of "50 < 49" is "50 ≥ 49": $\sqrt{50} < 7$ and $50 \geq 49$. Since $7^2 = 49 < 50$, $\sqrt{50} > 7$: the hypothesis is false. The statement is vacuously true, and its negation, which needs $\sqrt{50} < 7$, is false.

(d) The pool is heated and the swim meet does not go ahead.

(e) The hypothesis "Naomi does not practice" stays as it is, and the negated conclusion "not (she does not pass)" is "she passes": Naomi does not practice and she passes her driving test.

3. Each sentence below is an or-statement. Rewrite it as an if-then sentence. In (d) and (e), n is an integer: also decide whether the sentence is true for every integer n, and if it is not, name an integer that makes it false.

(a) The bakery sells out by noon, or Hannah buys bread there after school.

(b) The bridge is closed, or the ferry does not run.

(c) Elena does not get a ticket, or she goes to the concert.

(d) n is odd, or n^2 is a multiple of 4.

(e) n is a multiple of 3, or $n^2 + 1$ is even.

ANSWER

(a) If the bakery does not sell out by noon, then Hannah buys bread there after school. (b) If the bridge is open, then the ferry does not run. (c) If Elena gets a ticket, then she goes to the concert. (d) If n is even, then n^2 is a multiple of 4; true for every integer. (e) If n is not a multiple of 3, then $n^2 + 1$ is even; false, for example $n = 2$.

SOLUTION

An or-statement $A \vee B$ is the or-form of "if not A, then B": negate the first part to make the hypothesis, and keep the second part as the conclusion.

(a) If the bakery does not sell out by noon, then Hannah buys bread there after school.

(b) "Not closed" is "open": if the bridge is open, then the ferry does not run.

(c) The first part is already a negation, and negating it removes the "not": if Elena gets a ticket, then she goes to the concert.

(d) If n is even, then n^2 is a multiple of 4. True: an even n is $2k$ for an integer k, and $(2k)^2 = 4k^2$.

(e) If n is not a multiple of 3, then $n^2 + 1$ is even. False: $n = 2$ is not a multiple of 3, and $n^2 + 1 = 5$ is odd. In the or-form, both parts fail at $n = 2$.

Each part has a second correct answer that starts from the other part, since the parts of an or can be listed in either order. For (a): if Hannah does not buy bread there after school, then the bakery sells out by noon. This is the contrapositive of the first answer.

4. Show each equivalence without a truth table. Begin by rewriting every conditional in its or-form. Besides that rewriting you may use the De Morgan laws, cancelling a double negation, reordering and regrouping the parts

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of an or, the fact that $A \vee A \equiv A$, and, in (c), the law $(A \vee C) \wedge (B \vee C) \equiv (A \wedge B) \vee C$.

(a) $P \to \neg P \equiv \neg P$. Then say in words what this tells you about a statement that implies its own negation.

(b) $(P \wedge Q) \to R \equiv P \to (Q \to R)$

(c) $(P \to R) \wedge (Q \to R) \equiv (P \vee Q) \to R$

ANSWER

(a) $P \to \neg P \equiv \neg P \vee \neg P \equiv \neg P$: a statement that implies its own negation is false. (b) Both sides are $\neg P \vee \neg Q \vee R$. (c) Both sides are $(\neg P \wedge \neg Q) \vee R$.

SOLUTION

(a) The or-form of $P \to \neg P$ is $\neg P \vee \neg P$, and an or of a statement with itself is that statement:

$$P \to \neg P \ \equiv \ \neg P \vee \neg P \ \equiv \ \neg P$$

So whenever "if P then not P" is true, $\neg P$ is true: a statement that implies its own negation is false.

(b) Rewrite each side, using the De Morgan law for and on the left and the or-form twice on the right:

$$\begin{aligned}(P \wedge Q) \to R &\equiv \neg(P \wedge Q) \vee R \\ &\equiv \neg P \vee \neg Q \vee R\end{aligned}$$

$$\begin{aligned}P \to (Q \to R) &\equiv \neg P \vee (Q \to R) \\ &\equiv \neg P \vee (\neg Q \vee R) \\ &\equiv \neg P \vee \neg Q \vee R\end{aligned}$$

Both sides are equivalent to the same statement, so they are equivalent to each other: "if P and Q, then R" says the same as "if P, then, if Q, then R".

(c) On the left, rewrite both conditionals, then use the given law with $A = \neg P$, $B = \neg Q$ and $C = R$:

$$\begin{aligned}(P \to R) \wedge (Q \to R) &\equiv (\neg P \vee R) \wedge (\neg Q \vee R) \\ &\equiv (\neg P \wedge \neg Q) \vee R\end{aligned}$$

On the right, use the or-form and then the De Morgan law for or:

$$\begin{aligned}(P \vee Q) \to R &\equiv \neg(P \vee Q) \vee R \\ &\equiv (\neg P \wedge \neg Q) \vee R\end{aligned}$$

The two sides agree. In words: "if it rains, the game is off, and if it snows, the game is off" says the same as "if it rains or snows, the game is off".

5. Let n be an integer, and consider the statement: if n is odd, then $n^2 - 1$ is a multiple of 8.

(a) Rewrite the statement as an or-statement, without "if".

(b) Prove the statement by showing that one part of your or-statement holds for every integer n. (Hint: sort the integers by their remainder on division by 4.)

ANSWER

(a) n is even, or $n^2 - 1$ is a multiple of 8. (b) For $n = 4k$ or $4k + 2$, n is even. For $n = 4k + 1$, $n^2 - 1 = 8(2k^2 + k)$; for $n = 4k + 3$, $n^2 - 1 = 8(2k^2 + 3k + 1)$.

SOLUTION

(a) The statement is $P \to Q$ with P "n is odd" and Q "$n^2 - 1$ is a multiple of 8". Its or-form $\neg P \vee Q$ is: n is even, or $n^2 - 1$ is a multiple of 8.

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(b) Every integer leaves remainder 0, 1, 2 or 3 on division by 4, so $n = 4k$, $4k + 1$, $4k + 2$ or $4k + 3$ for some integer k. It is enough to show that one part of the or-statement holds in each case.

If $n = 4k$ or $n = 4k + 2 = 2(2k + 1)$, then n is even: the first part holds, and there is nothing to show.

If $n = 4k + 1$, then

$$n^2 - 1 = 16k^2 + 8k = 8(2k^2 + k)$$

and $2k^2 + k$ is an integer, so $n^2 - 1$ is a multiple of 8.

If $n = 4k + 3$, then

$$n^2 - 1 = 16k^2 + 24k + 8 = 8(2k^2 + 3k + 1)$$

and $2k^2 + 3k + 1$ is an integer, so $n^2 - 1$ is a multiple of 8.

One part of the or-statement holds in every case, so it is true for every integer n, and so is the conditional.

Check: $n = 7$ gives $48 = 8 \cdot 6$, and $n = 9$ gives $80 = 8 \cdot 10$.

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