Discrete Mathematics and Proof Unit 1 worksheet: And and or: conjunction and disjunction (answer key)

ANSWERS AT A GLANCE

1. (a) true (b) false (c) false (d) false (e) true (f) true
2. (a) $S \wedge \neg M$ (b) $\neg S \wedge \neg M$ (c) $\neg(S \wedge M)$ (d) $(S \vee M) \wedge \neg(S \wedge M)$, which is $S \oplus M$ (e) The trail is not snowy or the trail is muddy. (f) (a), (c) and (d) are true; (b) is false.
3. (a) $x > -3$ and $x < 7$ (b) $x \leq -5$ or $x \geq 3$ (c) $x < 6$ or $x > 6$ (d) $x = 0$ or $y = 0$ (e) $x = 0$ and $y = 0$ (f) ($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$)
4. (a) $S \vee (\neg S \wedge R)$ (b) The column for the rule is T, T, T, F (rows in the usual order). (c) $S \vee R$
5. (a) 7 rows (b) 9 rows (c) 9 rows
6. See the proof below.

WORKED SOLUTIONS

1. Suppose P is true, Q is false and R is true. Find the truth value of each statement.

(a) $Q \vee R$ (b) $P \wedge \neg R$ (c) $\neg P \vee Q$

(d) $(P \vee Q) \wedge (Q \vee \neg R)$ (e) $\neg(P \wedge Q) \wedge R$ (f) $(\neg P \vee \neg Q) \wedge (Q \vee R)$

ANSWER

(a) true (b) false (c) false (d) false (e) true (f) true

SOLUTION

(a) R is true, and an or with a true part is **true**.

(b) R is true, so $\neg R$ is false, and an and with a false part is **false**.

(c) $\neg P$ is false and Q is false. Both parts are false, so the or is **false**.

(d) $P \vee Q$ is true, but $Q \vee \neg R$ has both parts false (Q and $\neg R$), so it is false. The and has a false part: **false**.

(e) $P \wedge Q$ is false because Q is, so $\neg(P \wedge Q)$ is true. R is true too, so both parts of the and hold: **true**.

(f) $\neg Q$ is true, so $\neg P \vee \neg Q$ is true; R is true, so $Q \vee R$ is true. Both parts of the and hold: **true**.

2. A park posts a daily trail report. Let S be "the trail is snowy" and M be "the trail is muddy". Write each sentence in (a) to (d) in symbols.

(a) The trail is snowy, but it is not muddy.

(b) The trail is neither snowy nor muddy.

(c) The trail is not both snowy and muddy.

(d) The trail is snowy or muddy, but not both.

(e) Write $\neg S \vee M$ as an English sentence.

(f) On a day when the trail is snowy and not muddy, which of the sentences (a) to (d) are true?

ANSWER

(a) $S \wedge \neg M$ (b) $\neg S \wedge \neg M$ (c) $\neg(S \wedge M)$ (d) $(S \vee M) \wedge \neg(S \wedge M)$, which is $S \oplus M$ (e) The trail is not snowy or the trail is muddy. (f) (a), (c) and (d) are true; (b) is false.

SOLUTION

(a) In logic "but" means "and": it signals a contrast and adds no other condition. So the sentence is $S \wedge \neg M$.

(b) "Neither snowy nor muddy" says not snowy and not muddy: $\neg S \wedge \neg M$.

(c) "Not both" negates the conjunction as a whole: $\neg(S \wedge M)$. It is true whenever at least one of the two fails, so it is a different statement from (b).

Free sample from math.wiki for classroom use. Do not post. math.wiki

(d) "Snowy or muddy" is the inclusive $S \vee M$, and "but not both" strikes out the case where both hold: $(S \vee M) \wedge \neg(S \wedge M)$, which is the exclusive or $S \oplus M$.

(e) The negation belongs to S alone: "The trail is not snowy or the trail is muddy." It does not say "not (snowy or muddy)".

(f) Here S is true and M is false. In (a), S and $\neg M$ are both true, so it is true. In (b), $\neg S$ is false, so it is false. In (c), $S \wedge M$ is false, so its negation is true. In (d), $S \vee M$ is true and $\neg(S \wedge M)$ is true, so it is true. So (a), (c) and (d) are true.

3. Here x and y are real numbers. Write each condition as a sentence with the same meaning whose parts are joined by "and" or "or". Use no absolute value bars, no $\neq$ sign and no chained inequality such as $a < x < b$.

(a) $|x - 2| < 5$

(b) $|x + 1| \geq 4$

(c) $x \neq 6$

(d) $xy = 0$

(e) $x^2 + y^2 = 0$

(f) $xy > 0$

ANSWER

(a) $x > -3$ and $x < 7$ (b) $x \leq -5$ or $x \geq 3$ (c) $x < 6$ or $x > 6$ (d) $x = 0$ or $y = 0$ (e) $x = 0$ and $y = 0$ (f) ($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$)

SOLUTION

(a) $|x - 2| < 5$ says $x - 2$ lies strictly between −5 and 5, so both "$x - 2 > -5$" and "$x - 2 < 5$" must hold: $x > -3$ and $x < 7$.

(b) $|x + 1| \geq 4$ holds when $x + 1$ is at least 4 above zero or at least 4 below it: "$x + 1 \leq -4$ or $x + 1 \geq 4$", that is, $x \leq -5$ or $x \geq 3$. No number makes both parts true, but that does not matter: one true part is enough.

(c) A real number other than 6 is less than 6 or greater than 6: $x < 6$ or $x > 6$.

(d) A product is zero exactly when at least one factor is zero (the zero product property), and "at least one" is inclusive or: $x = 0$ or $y = 0$.

(e) Squares of real numbers are never negative, so a sum of two squares is 0 only when both squares are 0: $x = 0$ and $y = 0$. "Or" would be wrong here: $x = 0$, $y = 3$ makes the sum 9.

(f) A product of two real numbers is positive exactly when neither is zero and they have the same sign. "Same sign" is two cases, each a conjunction: ($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$).

4. A film streaming service has this rule: a member may watch a film when the member has a subscription, or when the member has no subscription and has rented that film. Let S be "the member has a subscription" and R be "the member has rented the film".

(a) Write the rule as a statement in S and R.

(b) Make its truth table, with a helper column for each piece.

(c) Find a shorter statement in S and R with the same column, and explain in a sentence or two why the words "has no subscription and" can be dropped from the rule.

ANSWER

(a) $S \vee (\neg S \wedge R)$ (b) The column for the rule is T, T, T, F (rows in the usual order). (c) $S \vee R$

SOLUTION

(a) The rule is an or of two pieces, the second itself an and: $S \vee (\neg S \wedge R)$.

Free sample from math.wiki for classroom use. Do not post. math.wiki

(b)

Row	S	R	$\neg S$	$\neg S \wedge R$	$S \vee (\neg S \wedge R)$
1	T	T	F	F	T
2	T	F	F	F	T
3	F	T	T	T	T
4	F	F	T	F	F

(c) The column T, T, T, F is false only in row 4, which is the column of $S \vee R$. The words can go because the second piece only matters when S is false: when S is true the or is already true. And when S is false, $\neg S$ is true, so $\neg S \wedge R$ has the same value as R. So the rule says the same as "the member has a subscription or has rented the film".

5. Each statement below is made from the four letters P, Q, R, S, so its truth table has 16 rows. Without writing out the table, find how many rows make each statement true, and explain how you counted.

(a) $(P \vee Q \vee R) \wedge S$

(b) $(P \wedge Q \wedge R) \vee S$

(c) $(P \vee Q) \wedge (R \vee S)$

ANSWER

(a) 7 rows (b) 9 rows (c) 9 rows

SOLUTION

(a) An and is true only when both parts are. S must be true. Of the $2^3 = 8$ settings of P, Q, R, the or fails only in the one where all three are false, so 8 − 1 = 7 settings make it true. Each of these with S true is one row: 7 rows.

(b) An or is false only when both parts are false: S false and $P \wedge Q \wedge R$ false. The conjunction is false in 8 − 1 = 7 of the settings of P, Q, R, so 7 rows are false, and 16 − 7 = 9 rows are true.

(c) $P \vee Q$ is true for 3 of the 4 settings of P and Q, and $R \vee S$ for 3 of the 4 settings of R and S. The and needs both, and a setting of the first pair combines with any setting of the second, so 3 × 3 = 9 rows.

6. Prove that for every real number x, the statement "$x \geq 1$ or $(x-1)(x-2) > 0$" is true.

ANSWER

If $x \geq 1$ the first part holds. Otherwise $x < 1$, so $x - 1$ and $x - 2$ are both negative and $(x-1)(x-2) > 0$. One part always holds.

SOLUTION

Proof. Let x be any real number. If $x \geq 1$, the first part is true, and an or with a true part is true. So it is enough to show that whenever the first part fails, the second part holds.

Suppose $x \geq 1$ is false, so $x < 1$. Then $x - 1 < 0$, and $x - 2 < x - 1 < 0$. Both factors of $(x-1)(x-2)$ are negative, and a product of two negative numbers is positive, so $(x-1)(x-2) > 0$: the second part is true.

So for every real number x at least one part holds, and the disjunction is true. ■

Check. $x = 0$: (−1)(−2) = 2 > 0. Neither part alone holds for every x: $x = 0$ makes "$x \geq 1$" false, and $x = \frac{3}{2}$ gives $\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{4}$, so the second part fails there. At $x = 3$ both parts are true, which the inclusive or allows.

Free sample from math.wiki for classroom use. Do not post. math.wiki