Discrete Mathematics and Proof Unit 1 worksheet: A conditional and its contrapositive (answer key)

ANSWERS AT A GLANCE

1. (a) $Q \to P$ (b) $R \to \neg Q$ (c) $\neg R \to (\neg P \vee \neg Q)$ (d) $(\neg Q \vee R) \to P$ (e) $(\neg R \vee \neg S) \to (\neg P \wedge Q)$
2. (a) If no two sides of a triangle are equal, then no two of its angles are equal. (b) If Elena does not reach school before 8 a.m., then the train is not on time or the bus is late. (c) For every integer n: if n is odd, then n is not a multiple of 6. (d) If a soccer team does not win a match, then it scores at most 2 goals in it.
3. (a) Contrapositive: if x and y are both rational, then xy is rational. True, so (a) is true. (b) Contrapositive: if x is rational or y is rational, then $x + y$ is rational. False, so (b) is false: $x = 1$, $y = \sqrt{2}$. (c) No: $x = y = \sqrt{2}$ gives $xy = 2$.
4. (a) Converse: if the bus arrived late, then it rained. Inverse: if it did not rain, then the bus was not late. Contrapositive: if the bus was not late, then it did not rain. (b) The claim and the contrapositive hold for every trip; the converse and the inverse are each broken by trips 3 and 6. (c) It rained and the bus was not late; it breaks the contrapositive only.
5. See the proof below.
6. See the proof below.

WORKED SOLUTIONS

1. Write the contrapositive of each statement in symbols. Simplify it so that every $\neg$ sits on a single letter and no letter carries two of them.

   (a) $\neg P \to \neg Q$

   (b) $Q \to \neg R$

   (c) $(P \wedge Q) \to R$

   (d) $\neg P \to (Q \wedge \neg R)$

   (e) $(P \vee \neg Q) \to (R \wedge S)$

   ANSWER

   (a) $Q \to P$ (b) $R \to \neg Q$ (c) $\neg R \to (\neg P \vee \neg Q)$ (d) $(\neg Q \vee R) \to P$ (e) $(\neg R \vee \neg S) \to (\neg P \wedge Q)$

   SOLUTION

   The contrapositive of $A \to B$ is $\neg B \to \neg A$: the conclusion, negated, becomes the hypothesis, and the hypothesis, negated, becomes the conclusion.

   (a) $\neg\neg Q \to \neg\neg P$, and a double negation cancels: $Q \to P$.

   (b) $\neg\neg R \to \neg Q$, which is $R \to \neg Q$.

   (c) $\neg R \to \neg(P \wedge Q)$. By the De Morgan laws, $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$, so the contrapositive is $\neg R \to (\neg P \vee \neg Q)$.

   (d) $\neg(Q \wedge \neg R) \to \neg\neg P$. The De Morgan laws turn the and into an or, and the $\neg\neg R$ cancels: $(\neg Q \vee R) \to P$.

   (e) $\neg(R \wedge S) \to \neg(P \vee \neg Q)$, which is $(\neg R \vee \neg S) \to (\neg P \wedge Q)$. Negating each part flips its connective, by the De Morgan laws, and it is the parts that are negated, never the arrow.

2. Write the contrapositive of each statement as a plain sentence. Do not begin a part with "it is not the case". You do not have to decide whether the statements are true.

   (a) If a triangle has two equal angles, then it has two equal sides.

   (b) If the train is on time and the bus is not late, then Elena reaches school before 8 a.m.

   (c) Every multiple of 6 is even. (First write it as an if-then statement about an integer n.)

   (d) If a soccer team scores at least 3 goals in a match, then it wins the match.

   ANSWER

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(a) If no two sides of a triangle are equal, then no two of its angles are equal. (b) If Elena does not reach school before 8 a.m., then the train is not on time or the bus is late. (c) For every integer n: if n is odd, then n is not a multiple of 6. (d) If a soccer team does not win a match, then it scores at most 2 goals in it.

SOLUTION

In each part, swap the hypothesis and the conclusion and negate both.

(a) The negation of "has two equal sides" is "has no two equal sides", and likewise for the angles: if no two sides of a triangle are equal, then no two of its angles are equal.

(b) The conclusion negated is "Elena does not reach school before 8 a.m." The hypothesis is an and, so its negation is an or (the De Morgan laws), and "not (the bus is not late)" is "the bus is late": if Elena does not reach school before 8 a.m., then the train is not on time or the bus is late.

(c) As a conditional: for every integer n, if n is a multiple of 6, then n is even. The words "for every integer n" set the scene and are not negated. For an integer, "not even" is "odd": for every integer n, if n is odd, then n is not a multiple of 6.

(d) The negation of "scores at least 3 goals" is "scores fewer than 3 goals", that is, at most 2, since goals are whole numbers: if a soccer team does not win a match, then it scores at most 2 goals in it.

3. Here x and y are real numbers. In (a) and (b), write the contrapositive and use it to decide whether the statement is true. Give a counterexample to any false statement.

(a) If xy is irrational, then x is irrational or y is irrational.

(b) If $x + y$ is irrational, then x is irrational and y is irrational.

(c) Is the converse of the statement in (a) true? Explain.

ANSWER

(a) Contrapositive: if x and y are both rational, then xy is rational. True, so (a) is true. (b) Contrapositive: if x is rational or y is rational, then $x + y$ is rational. False, so (b) is false: $x = 1$, $y = \sqrt{2}$. (c) No: $x = y = \sqrt{2}$ gives $xy = 2$.

SOLUTION

(a) The conclusion is an or, so its negation is an and (the De Morgan laws): if x is rational and y is rational, then xy is rational. This is true: write $x = \dfrac{a}{b}$ and $y = \dfrac{c}{d}$ with integers a, b, c, d and $b, d \neq 0$. Then $xy = \dfrac{ac}{bd}$, a ratio of integers with $bd \neq 0$, so xy is rational. The contrapositive is true, so the statement is true.

(b) Now the conclusion is an and, so its negation is an or: if x is rational or y is rational, then $x + y$ is rational. This is false. Take $x = 1$ and $y = \sqrt{2}$: x is rational, but $x + y = 1 + \sqrt{2}$ is irrational (if it were a rational number r, then $\sqrt{2} = r - 1$ would be rational). The contrapositive is false, so the statement is false, and the same pair breaks it: $x + y$ is irrational, yet x is rational.

(c) The converse is "if x is irrational or y is irrational, then xy is irrational". It is false: $x = y = \sqrt{2}$ are both irrational, but $xy = 2$ is rational. The truth of (a) says nothing about its converse.

4. A bus company's log for eight trips records whether it rained during the trip and whether the bus arrived late.

Trip	1	2	3	4	5	6	7	8
Rained	yes	no	no	yes	no	no	yes	no
Late	yes	no	yes	yes	no	yes	yes	no

Consider the claim about these trips: "If it rained during a trip, then the bus arrived late."

(a) Write the converse, the inverse and the contrapositive of the claim.

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(b) For each of the four statements, decide whether it holds for every trip in the log, and name every trip that breaks it.

(c) A ninth trip is added to the log, and the claim no longer holds for all nine trips. What happened on the ninth trip? Which of the other three statements does that trip break?

ANSWER

(a) Converse: if the bus arrived late, then it rained. Inverse: if it did not rain, then the bus was not late. Contrapositive: if the bus was not late, then it did not rain. (b) The claim and the contrapositive hold for every trip; the converse and the inverse are each broken by trips 3 and 6. (c) It rained and the bus was not late; it breaks the contrapositive only.

SOLUTION

(a) Let P be "it rained during the trip" and Q be "the bus arrived late". The converse $Q \to P$: if the bus arrived late, then it rained. The inverse $\neg P \to \neg Q$: if it did not rain, then the bus was not late. The contrapositive $\neg Q \to \neg P$: if the bus was not late, then it did not rain.

(b) Each statement is broken only by a trip where its hypothesis is true and its conclusion false.

- Claim: broken by a trip with rain and no delay. The rainy trips 1, 4 and 7 were all late: it holds.
- Contrapositive: broken by a trip that was not late but had rain. The trips that were not late, 2, 5 and 8, all had no rain: it holds, as it must, since it rules out the same case as the claim.
- Converse: broken by a late trip with no rain: trips 3 and 6.
- Inverse: broken by a trip with no rain that was late: again trips 3 and 6. The converse and the inverse rule out the same case, so the same trips break both.

(c) The claim fails only on a trip with rain where the bus was not late, so that is what happened. That trip has a true hypothesis and a false conclusion for the contrapositive ("not late", yet it rained), so it breaks the contrapositive too. It was not late, so the converse has a false hypothesis there; it rained, so the inverse has a false hypothesis there. It breaks the contrapositive only.

5. Let x be a real number.

(a) Prove, by proving the contrapositive, that if x is irrational, then $5x - 2$ is irrational.

(b) Show that the statement "if x is irrational, then x^2 is irrational" is false.

ANSWER

(a) Contrapositive: if $5x - 2$ is rational, then x is rational; from $5x - 2 = r$, $x = \dfrac{r+2}{5}$ is rational. (b) $x = \sqrt{2}$ is irrational, but $x^2 = 2$ is rational.

SOLUTION

(a) For a real number, "not irrational" is "rational", so the contrapositive is: if $5x - 2$ is rational, then x is rational.

Suppose $5x - 2$ is rational, say $5x - 2 = r$ where $r = \dfrac{a}{b}$ for integers a and b with $b \neq 0$. Then

$$x = \frac{r+2}{5} = \frac{a+2b}{5b}$$

a ratio of integers with $5b \neq 0$, so x is rational.

The contrapositive holds for every real number x, and a conditional is logically equivalent to its contrapositive, so: if x is irrational, then $5x - 2$ is irrational.

(b) A conditional is false when some value makes its hypothesis true and its conclusion false. $x = \sqrt{2}$ is irrational, but $x^2 = 2$ is rational, so the statement is false. (Its contrapositive, "if x^2 is rational, then x is

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rational", fails at the same x.)

6. Malik wants to prove that for every integer n, if $n^2 + 1$ is even, then n is odd. He writes:

> The contrapositive is: if n is odd, then $n^2 + 1$ is even. Suppose n is odd, so $n = 2k + 1$ for an integer k. Then $n^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$, which is even. So the statement is proved.

(a) Which of the converse, the inverse and the contrapositive of the statement did Malik actually prove?

(b) Does his work prove the statement? Explain.

(c) Write the true contrapositive of the statement, and prove it.

ANSWER

(a) The converse. (b) No: a conditional is not logically equivalent to its converse. (c) If n is even, then $n^2 + 1$ is odd; with $n = 2k$, $n^2 + 1 = 2(2k^2) + 1$.

SOLUTION

(a) The statement is $P \to Q$ with P "$n^2 + 1$ is even" and Q "n is odd". Malik proved "if n is odd, then $n^2 + 1$ is even", which is $Q \to P$, the converse. He swapped the parts but did not negate them.

(b) No. His algebra is right, so the converse is true, but the converse is not logically equivalent to the statement: $P \to Q$ and $Q \to P$ disagree whenever exactly one of P and Q is true. Knowing the converse leaves the statement unproved.

(c) The contrapositive $\neg Q \to \neg P$ is "if n is not odd, then $n^2 + 1$ is not even", that is, for an integer, "if n is even, then $n^2 + 1$ is odd".

Proof. Suppose n is even, so $n = 2k$ for an integer k. Then

$$n^2 + 1 = 4k^2 + 1 = 2(2k^2) + 1$$

and $2k^2$ is an integer, so $n^2 + 1$ is odd. The contrapositive holds for every integer, so the statement does too: if $n^2 + 1$ is even, then n is odd.

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