Discrete Mathematics and Proof Unit 1 worksheet: De Morgan's laws for statements (answer key)

ANSWERS AT A GLANCE

1. (a) The museum is not open on Monday or admission is not free. (b) The bus was not late and the driver did not take a different route. (c) The soup is salty or it is spicy. (d) The dough has not risen or the oven is not hot or the tray is not greased.
2. (a) $\neg P \lor Q \lor \neg R$ (b) $(\neg P \land \neg Q) \lor R$ (c) $P \land Q \land \neg R$ (d) $(\neg P \lor Q) \land (P \lor \neg R)$
3. (a) wrong: it disagrees only at $x = -3$ (b) wrong: it is true for no x, so it disagrees at every $x \leq -3$ and every $x > 5$ (c) wrong: it disagrees only at $x = 5$ (d) correct
4. (a) turn down: $T \lor S \lor E$; accept: $\neg T \land \neg S \land \neg E$, that is, the username is free, the password has at least 8 characters and the email address is not yet in use (b) for example, a taken username with a long enough password and a new email address; the programmer's condition is the negation of $T \land S \land E$, so it turns down only an account with all three problems at once
5. (a) equivalent (b) not equivalent: they disagree in rows 6 and 7 (c) the correct statement is $\neg P \land (\neg Q \lor \neg R)$; the student negated the inner "and" $Q \land R$ without flipping it to "or"
6. (a) $\big(P \lor (\neg Q \land R)\big) \land (\neg R \lor Q)$ (b) S is true and the negation is false (c) each use of a De Morgan law at the outermost connective swaps that connective and moves the $\neg$ onto both of its parts; repeating this reaches every connective once and every letter once, where $\neg$ on a bare letter adds a $\neg$ and $\neg$ on a negated letter cancels by double negation

WORKED SOLUTIONS

1. Write the negation of each sentence as a plain sentence. Do not begin with "it is not the case that".

   (a) The museum is open on Monday and admission is free.

   (b) The bus was late or the driver took a different route.

   (c) The soup is neither salty nor spicy.

   (d) The dough has risen, the oven is hot and the tray is greased.

   ANSWER

   (a) The museum is not open on Monday or admission is not free. (b) The bus was not late and the driver did not take a different route. (c) The soup is salty or it is spicy. (d) The dough has not risen or the oven is not hot or the tray is not greased.

   SOLUTION

   (a) The sentence is an "and". By the "and" law its negation is the "or" of the two negations: the museum is not open on Monday or admission is not free. One failing part is enough to break the original.

   (b) The sentence is an "or". By the "or" law its negation is the "and" of the two negations: the bus was not late and the driver did not take a different route. Both parts have to fail.

   (c) "Neither salty nor spicy" says "not salty and not spicy", an "and" of two negations. Its negation by the "and" law is "not not salty or not not spicy", and the double negations cancel: the soup is salty or it is spicy (or both).

   (d) A three-part "and". Applying the "and" law twice gives the three-part "or" of the negations: the dough has not risen or the oven is not hot or the tray is not greased.

2. Rewrite each statement so that every $\neg$ sits on a single letter, simplifying any double negation. Show each use of a De Morgan law.

   (a) $\neg(P \land \neg Q \land R)$

   (b) $\neg\big((P \lor Q) \land \neg R\big)$

   (c) $\neg\big(\neg(P \land Q) \lor R\big)$

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(d) $\neg\big((P \wedge \neg Q) \vee (\neg P \wedge R)\big)$

ANSWER

(a) $\neg P \vee Q \vee \neg R$ (b) $(\neg P \wedge \neg Q) \vee R$ (c) $P \wedge Q \wedge \neg R$ (d) $(\neg P \vee Q) \wedge (P \vee \neg R)$

SOLUTION

(a) The "and" law for three parts gives $\neg P \vee \neg\neg Q \vee \neg R$, and $\neg\neg Q$ is Q: $\neg P \vee Q \vee \neg R$.

(b) The "and" law on the outside: $\neg(P \vee Q) \vee \neg\neg R$. Then the "or" law on the first part and double negation on the second: $(\neg P \wedge \neg Q) \vee R$.

(c) The "or" law on the outside: $\neg\neg(P \wedge Q) \wedge \neg R$. Double negation: $(P \wedge Q) \wedge \neg R$, which is $P \wedge Q \wedge \neg R$. The inner $P \wedge Q$ ends up with no $\neg$ on it, so no second law is needed there.

(d) The "or" law on the outside: $\neg(P \wedge \neg Q) \wedge \neg(\neg P \wedge R)$. The "and" law on each part: $(\neg P \vee \neg\neg Q) \wedge (\neg\neg P \vee \neg R)$. Double negations: $(\neg P \vee Q) \wedge (P \vee \neg R)$.

3. The condition $-3 < x \leq 5$ means "$x > -3$ and $x \leq 5$". Four students wrote its negation as follows.

(a) $x < -3$ or $x > 5$

(b) $x \leq -3$ and $x > 5$

(c) $x \leq -3$ or $x \geq 5$

(d) $x \leq -3$ or $x > 5$

Say which are correct. For each one that is wrong, find every real number x at which it disagrees with the true negation.

ANSWER

(a) wrong: it disagrees only at $x = -3$ (b) wrong: it is true for no x, so it disagrees at every $x \leq -3$ and every $x > 5$ (c) wrong: it disagrees only at $x = 5$ (d) correct

SOLUTION

By the "and" law, the negation of "$x > -3$ and $x \leq 5$" is "$x \leq -3$ or $x > 5$": the strict $>$ flips to $\leq$, and $\leq$ flips to the strict $>$. That is answer (d), so (d) is correct.

(a) Here $x > -3$ was negated as $x < -3$. At $x = -3$ the original is false, since $-3 > -3$ fails, so the true negation is true there; but $-3 < -3$ and $-3 > 5$ are both false, so (a) is false. Everywhere else the two agree. They disagree only at $x = -3$.

(b) The connective was not flipped. No number is both ≤ -3 and > 5, so (b) is false for every x. It disagrees with the true negation wherever that is true: at every $x \leq -3$ and every $x > 5$.

(c) Here $x \leq 5$ was negated as $x \geq 5$. At $x = 5$ the original is true, since $-3 < 5 \leq 5$, so the true negation is false there; but (c) is true, since $5 \geq 5$. They disagree only at $x = 5$.

4. A website turns down a new account when the username is already taken, or the password is shorter than 8 characters, or the email address is already in use. Let T be "the username is already taken", S be "the password is shorter than 8 characters" and E be "the email address is already in use".

(a) Write in symbols the condition for turning an account down. Then use the De Morgan laws to write the condition for accepting it, with every $\neg$ on a single letter, and say it in words.

(b) The site's programmer coded the accept condition as $\neg T \vee \neg S \vee \neg E$. Describe an account this lets through that the site should turn down, and say which statement the programmer's condition is really the negation of.

ANSWER

(a) turn down: $T \vee S \vee E$; accept: $\neg T \wedge \neg S \wedge \neg E$, that is, the username is free, the password has at least 8 characters and the email address is not yet in use (b) for example, a taken username with a long

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enough password and a new email address; the programmer's condition is the negation of $T \wedge S \wedge E$, so it turns down only an account with all three problems at once

SOLUTION

(a) The account is turned down when $T \vee S \vee E$. It is accepted exactly when this fails, $\neg(T \vee S \vee E)$, and the "or" law for three parts gives $\neg T \wedge \neg S \wedge \neg E$: the username is free, the password has at least 8 characters, and the email address is not yet in use. All three checks must pass.

(b) Take an account whose username is taken but whose password is long enough and whose email address is new: T is true, S and E are false. Then $\neg S$ is true, so $\neg T \vee \neg S \vee \neg E$ is true and the account gets through, although T alone should turn it down. By the "and" law, $\neg T \vee \neg S \vee \neg E$ is $\neg(T \wedge S \wedge E)$: it only says "not all three problems at once". The programmer negated each piece but forgot to flip the connective.

5. Use a truth table (letters P, Q, R in the fixed order) to decide whether the statements in each pair are logically equivalent. When they are not, name every row where they disagree.

(a) $\neg\big(P \wedge (Q \vee R)\big)$ and $\neg P \vee (\neg Q \wedge \neg R)$

(b) $\neg\big(P \vee (Q \wedge R)\big)$ and $\neg P \wedge \neg Q \wedge \neg R$

(c) The pair that is not equivalent came from a student's attempt to push the $\neg$ inside. Find the mistake, and write the correct statement with every $\neg$ on a single letter.

ANSWER

(a) equivalent (b) not equivalent: they disagree in rows 6 and 7 (c) the correct statement is $\neg P \wedge (\neg Q \vee \neg R)$; the student negated the inner "and" $Q \wedge R$ without flipping it to "or"

SOLUTION

(a)

Row	P	Q	R	$Q \vee R$	$\neg\big(P \wedge (Q \vee R)\big)$	$\neg Q \wedge \neg R$	$\neg P \vee (\neg Q \wedge \neg R)$
1	T	T	T	T	F	F	F
2	T	T	F	T	F	F	F
3	T	F	T	T	F	F	F
4	T	F	F	F	T	T	T
5	F	T	T	T	T	F	T
6	F	T	F	T	T	F	T
7	F	F	T	T	T	F	T
8	F	F	F	F	T	T	T

The last two columns agree in all eight rows, so the statements are equivalent. This is the "and" law on the outside, then the "or" law on $\neg(Q \vee R)$.

(b)

Row	P	Q	R	$Q \wedge R$	$\neg\big(P \vee (Q \wedge R)\big)$	$\neg P \wedge \neg Q \wedge \neg R$
1	T	T	T	T	F	F
2	T	T	F	F	F	F
3	T	F	T	F	F	F
4	T	F	F	F	F	F
5	F	T	T	T	F	F
6	F	T	F	F	T	F
7	F	F	T	F	T	F
8	F	F	F	F	T	T

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The last two columns disagree in rows 6 and 7, so the statements are not equivalent.

(c) The "or" law on the outside is right: $\neg\big(P \vee (Q \wedge R)\big) \equiv \neg P \wedge \neg(Q \wedge R)$. The mistake is the next step: $\neg(Q \wedge R)$ needs the "and" law, which gives $\neg Q \vee \neg R$, not $\neg Q \wedge \neg R$. The correct statement is $\neg P \wedge (\neg Q \vee \neg R)$. Rows 6 and 7, where exactly one of Q, R is true, are where the student's version fails.

6. A statement uses only the connectives $\wedge$ and $\vee$, and every $\neg$ in it sits on a single letter. A classmate proposes a shortcut for writing its negation: swap every $\wedge$ with $\vee$ and every $\vee$ with $\wedge$, put a $\neg$ on every letter that has none, and remove the $\neg$ from every letter that has one.

Let S be $\big(\neg P \wedge (Q \vee \neg R)\big) \vee (R \wedge \neg Q)$.

(a) Use the shortcut to write the negation of S. Then find $\neg S$ again by applying the De Morgan laws one connective at a time, and check that the two agree.

(b) Find the truth value of S and of your answer to (a) when P is true, Q is false and R is true.

(c) Explain why the shortcut works for every statement of this kind.

ANSWER

(a) $\big(P \vee (\neg Q \wedge R)\big) \wedge (\neg R \vee Q)$ (b) S is true and the negation is false (c) each use of a De Morgan law at the outermost connective swaps that connective and moves the $\neg$ onto both of its parts; repeating this reaches every connective once and every letter once, where $\neg$ on a bare letter adds a $\neg$ and $\neg$ on a negated letter cancels by double negation

SOLUTION

(a) The shortcut swaps the three connectives and flips the five letters: $\big(P \vee (\neg Q \wedge R)\big) \wedge (\neg R \vee Q)$.

By the laws: the outer connective is $\vee$, so the "or" law gives $\neg\big(\neg P \wedge (Q \vee \neg R)\big) \wedge \neg(R \wedge \neg Q)$. The "and" law on each part gives $\big(\neg\neg P \vee \neg(Q \vee \neg R)\big) \wedge (\neg R \vee \neg\neg Q)$. The "or" law on $\neg(Q \vee \neg R)$ gives $\neg Q \wedge \neg\neg R$. Removing the double negations leaves $\big(P \vee (\neg Q \wedge R)\big) \wedge (\neg R \vee Q)$, the same as the shortcut.

(b) S: $\neg P$ is false, so the first part is false; $R \wedge \neg Q$ is true and true, which is true; so S is true. The negation: $P \vee (\neg Q \wedge R)$ is true, but $\neg R \vee Q$ is false or false, which is false; so the negation is false, the opposite of S, as it must be.

(c) Negating a statement whose outer connective is $\wedge$ or $\vee$ uses one De Morgan law: that connective swaps, and the $\neg$ moves onto each of its two parts. Each part is again a statement of the same kind, so the same step repeats inside it. Every connective is the outer connective of exactly one part at some stage, so every connective swaps exactly once. The $\neg$ stops only when it reaches a letter: on a bare letter P it leaves $\neg P$, and on a negated letter $\neg P$ it leaves $\neg\neg P$, which is P. So every letter flips exactly once, which is the shortcut.

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