Discrete Mathematics and Proof Unit 1 worksheet: Tautologies and contradictions (answer key)

ANSWERS AT A GLANCE

1. (a) tautology (b) contradiction (c) neither: T in rows 1 and 4, F in rows 2 and 3 (d) tautology (e) neither: T in rows 1, 3 and 4, F in row 2 (f) only the statement in (b)
2. (a) tautology: no row makes it false (b) contradiction: no row makes it true (c) not a tautology: false only in row 3
3. (a) last column T, T, T, T (b) $(R \vee S) \to (\neg R \to (R \vee S))$ (c) $P \to (Q \to (R \vee S))$; false when P is true, Q is true, R is false and S is false
4. (a) An "and" is T only when both parts are, so on every row both A and B are T. (b) For example $A = P$ and $B = \neg P$. (c) An "or" is F only when both parts are, so on every row both A and B are F.
5. (a) $\big((W \vee R) \wedge \neg W\big) \to R$ and $\big((W \vee R) \wedge W\big) \to \neg R$ (b) The first is a tautology: argument 1 is valid. The second is false in row 1: argument 2 is invalid; row 1 is a day that is both windy and rainy.
6. (a) equivalent: the biconditional is T in all 4 rows (b) not equivalent: the biconditional is false in rows 4 and 6

WORKED SOLUTIONS

1. Classify each statement as a tautology, a contradiction or neither. When it is neither, name the rows (letters P, Q in the fixed order) where it is T and the rows where it is F.

   (a) $\neg P \to (P \to Q)$

   (b) $(P \leftrightarrow Q) \wedge (\neg P \wedge Q)$

   (c) $(P \wedge Q) \leftrightarrow (P \vee Q)$

   (d) $(P \to Q) \vee (P \to \neg Q)$

   (e) $\neg(P \wedge Q) \to \neg P$

   (f) For which of the statements in (a) to (e) is the negation a tautology? Answer without building any new table.

   ANSWER

   (a) tautology (b) contradiction (c) neither: T in rows 1 and 4, F in rows 2 and 3 (d) tautology (e) neither: T in rows 1, 3 and 4, F in row 2 (f) only the statement in (b)

   SOLUTION

   (f) Negation flips every entry of a column, so the negation is a tautology exactly when the statement itself is F on every row, that is, a contradiction. Of (a) to (e), only (b) is a contradiction. The negations of (a) and (d) are contradictions, and those of (c) and (e) are neither.

   The tables for (a) to (e) follow.

   (a) Every row ends in T: a tautology. In rows 1 and 2 the hypothesis $\neg P$ is F; in rows 3 and 4, P is F, so $P \to Q$ is T.

Row	P	Q	$\neg P$	$P \to Q$	$\neg P \to (P \to Q)$
1	T	T	F	T	T
2	T	F	F	F	T
3	F	T	T	T	T
4	F	F	T	T	T

   (b) Every row ends in F: a contradiction. $\neg P \wedge Q$ is T only in row 3, where P and Q disagree, so $P \leftrightarrow Q$ is F there.

Row	P	Q	$P \leftrightarrow Q$	$\neg P \wedge Q$	$(P \leftrightarrow Q) \wedge (\neg P \wedge Q)$
1	T	T	T	F	F
2	T	F	F	F	F

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Row	P	Q	$P \leftrightarrow Q$	$\neg P \wedge Q$	$(P \leftrightarrow Q) \wedge (\neg P \wedge Q)$
3	F	T	F	T	F
4	F	F	T	F	F

(c) T in rows 1 and 4, F in rows 2 and 3: neither.

Row	P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q) \leftrightarrow (P \vee Q)$
1	T	T	T	T	T
2	T	F	F	T	F
3	F	T	F	T	F
4	F	F	F	F	T

(d) Every row ends in T: a tautology. The two parts are F only in row 2 and row 1 respectively, so in every row at least one of them is T.

Row	P	Q	$P \to Q$	$P \to \neg Q$	$(P \to Q) \vee (P \to \neg Q)$
1	T	T	T	F	T
2	T	F	F	T	T
3	F	T	T	T	T
4	F	F	T	T	T

(e) F only in row 2, where the hypothesis $\neg(P \wedge Q)$ is T and $\neg P$ is F: neither.

Row	P	Q	$\neg(P \wedge Q)$	$\neg P$	$\neg(P \wedge Q) \to \neg P$
1	T	T	F	F	T
2	T	F	T	F	F
3	F	T	T	T	T
4	F	F	T	T	T

2. Decide each part without writing all eight rows of the table (letters P, Q, R in the fixed order).

(a) Show that $\big(P \wedge (Q \vee R)\big) \to \big((P \wedge Q) \vee R\big)$ is a tautology by trying to make it false.

(b) Show that $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \wedge (P \wedge \neg R)$ is a contradiction by trying to make it true.

(c) Is $\big(P \to (Q \vee R)\big) \to (P \to Q)$ a tautology? If not, name every row in which it is false.

ANSWER

(a) tautology: no row makes it false (b) contradiction: no row makes it true (c) not a tautology: false only in row 3

SOLUTION

(a) A conditional is false only when its hypothesis is T and its conclusion is F. For the conclusion $(P \wedge Q) \vee R$ to be F, R is F and $P \wedge Q$ is F. For the hypothesis to be T, P is T and $Q \vee R$ is T; with R F, that forces Q T. Then $P \wedge Q$ is T, against what the conclusion needed. No row makes the statement false, so it is a tautology.

(b) An "and" is true only when every part is. The last part needs P T and R F. Then $P \leftrightarrow Q$ forces Q T, and $Q \leftrightarrow R$ forces R T. R would have to be T and F at once, so no row makes the statement true: a contradiction.

(c) Try to make it false. The conclusion $P \to Q$ must be F, so P is T and Q is F. The hypothesis $P \to (Q \vee R)$ must be T; with P T this needs $Q \vee R$ T, and with Q F that forces R T. The hunt succeeds: P T, Q F, R T, which is row 3, and every letter was forced, so no other row is false. Not a tautology.

3. (a) Use a truth table to show that $P \to (Q \to P)$ is a tautology.

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(b) Replace P by $R \vee S$ and Q by $\neg R$ at every place they occur. Write the new statement, and explain without a table why it is a tautology.

(c) Isaac replaces only the last P by $R \vee S$ and leaves the rest as it was. Write his statement, and show that it is not a tautology by giving truth values of its letters that make it false.

ANSWER

(a) last column T, T, T, T (b) $(R \vee S) \to (\neg R \to (R \vee S))$ (c) $P \to (Q \to (R \vee S))$; false when P is true, Q is true, R is false and S is false

SOLUTION

(a)

Row	P	Q	$Q \to P$	$P \to (Q \to P)$
1	T	T	T	T
2	T	F	T	T
3	F	T	F	T
4	F	F	T	T

Every row ends in T. In rows 3 and 4 the hypothesis P is F; in rows 1 and 2, $Q \to P$ has a true conclusion.

(b) $(R \vee S) \to (\neg R \to (R \vee S))$. On any row of R and S, the compound $R \vee S$ has some truth value and $\neg R$ has some truth value. That pair of values is one of the four rows of the table in (a), and every one of those rows ends in T. The table never looked at how the values were made, so the new statement is T on every row: a tautology.

(c) $P \to (Q \to (R \vee S))$. It is false only when P is T and $Q \to (R \vee S)$ is F, which needs Q T and $R \vee S$ F, so R F and S F. With P true, Q true, R false and S false the statement is false, so it is not a tautology. Replacing only one of the two Ps cut the tie between them that made the original always true.

4. A and B are statements built from the same letters.

(a) Explain why, if $A \wedge B$ is a tautology, then A and B are each tautologies.

(b) Give an example of statements A and B, built from the one letter P, for which $A \vee B$ is a tautology but neither A nor B is.

(c) Explain why, if $A \vee B$ is a contradiction, then A and B are each contradictions.

ANSWER

(a) An "and" is T only when both parts are, so on every row both A and B are T. (b) For example $A = P$ and $B = \neg P$. (c) An "or" is F only when both parts are, so on every row both A and B are F.

SOLUTION

(a) Take any row of the table. $A \wedge B$ is T on that row, because it is a tautology, and an "and" is T only when both of its parts are T. So A is T and B is T on that row. The row was any row, so A and B are T on every row: each is a tautology.

(b) Let $A = P$ and $B = \neg P$. Then $A \vee B$ is $P \vee \neg P$, which is T on both rows, a tautology. But $A = P$ is F on row 2 and $B = \neg P$ is F on row 1, so neither is a tautology. On each row a different part does the work, which an "or" allows and an "and" does not.

(c) Take any row. $A \vee B$ is F there, because it is a contradiction, and an "or" is F only when both of its parts are F. So A and B are both F on that row, and since the row was any row, each is F on every row: each is a contradiction.

5. A forecast says: "Tomorrow will be windy or rainy." Let W be "tomorrow is windy" and R be "tomorrow is rainy". Two listeners each add a premise and draw a conclusion.

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Argument 1: "Tomorrow will be windy or rainy. It will not be windy. So it will be rainy."

Argument 2: "Tomorrow will be windy or rainy. It will be windy. So it will not be rainy."

(a) Write each argument as one conditional.

(b) Decide whether each conditional is a tautology, and so whether each argument is valid. For an argument that is not valid, name the counterexample row (letters W, R in the fixed order) and say what kind of day it describes.

ANSWER

(a) $\big((W \vee R) \wedge \neg W\big) \to R$ and $\big((W \vee R) \wedge W\big) \to \neg R$ (b) The first is a tautology: argument 1 is valid. The second is false in row 1: argument 2 is invalid; row 1 is a day that is both windy and rainy.

SOLUTION

(a) The premises are joined by "and" before the arrow and the conclusion goes after it: $\big((W \vee R) \wedge \neg W\big) \to R$ and $\big((W \vee R) \wedge W\big) \to \neg R$.

(b) Argument 1:

Row	W	R	$W \vee R$	$(W \vee R) \wedge \neg W$	$\big((W \vee R) \wedge \neg W\big) \to R$
1	T	T	T	F	T
2	T	F	T	F	T
3	F	T	T	T	T
4	F	F	F	F	T

Every row ends in T, so the argument is valid: the only row with both premises true is row 3, and there R is T.

Argument 2:

Row	W	R	$(W \vee R) \wedge W$	$\big((W \vee R) \wedge W\big) \to \neg R$
1	T	T	T	F
2	T	F	T	T
3	F	T	F	T
4	F	F	F	T

Row 1 ends in F, so the conditional is not a tautology and the argument is invalid. Row 1 is a day that is both windy and rainy: the forecast is right, it is windy, and yet it rains. The forecast's "or" allows both.

6. Two statements are logically equivalent exactly when their biconditional is a tautology. Use this test to decide whether the statements in each pair are logically equivalent. When they are not, name every row (letters in the order P, Q, R) where the biconditional is false.

(a) $\neg(P \leftrightarrow Q)$ and $\neg P \leftrightarrow Q$

(b) $(P \wedge Q) \to R$ and $(P \to R) \wedge (Q \to R)$

ANSWER

(a) equivalent: the biconditional is T in all 4 rows (b) not equivalent: the biconditional is false in rows 4 and 6

SOLUTION

(a)

Row	P	Q	$P \leftrightarrow Q$	$\neg(P \leftrightarrow Q)$	$\neg P$	$\neg P \leftrightarrow Q$	$\neg(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow Q)$
1	T	T	T	F	F	F	T
2	T	F	F	T	F	T	T
3	F	T	F	T	T	T	T

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Row	P	Q	$P \leftrightarrow Q$	$\neg(P \leftrightarrow Q)$	$\neg P$	$\neg P \leftrightarrow Q$	$\neg(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow Q)$
4	F	F	T	F	T	F	T

The two columns agree in every row, so the biconditional is a tautology and the statements are equivalent. Both say that P and Q have different truth values.

(b)

Row	P	Q	R	$(P \wedge Q) \to R$	$(P \to R) \wedge (Q \to R)$	$\big((P \wedge Q) \to R\big) \leftrightarrow \big((P \to R) \wedge (Q \to R)\big)$
1	T	T	T	T	T	T
2	T	T	F	F	F	T
3	T	F	T	T	T	T
4	T	F	F	T	F	F
5	F	T	T	T	T	T
6	F	T	F	T	F	F
7	F	F	T	T	T	T
8	F	F	F	T	T	T

The biconditional is F in rows 4 and 6, so the statements are not equivalent. In row 4, P is T and Q and R are F: the left side is T because $P \wedge Q$ is F, while $P \to R$ fails, so the right side is F. Row 6 is the same with the roles of P and Q swapped. The left side needs both P and Q before it promises R; the right side promises R from either one alone.

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