The biconditional: necessary and sufficient

When do two statements always travel together, so that knowing either one tells you the other? And what do "necessary" and "sufficient" mean?

WORDS TO KNOW

Biconditional: "P if and only if Q", written $P \leftrightarrow Q$ (or $\iff$), sometimes shortened to "P iff Q".

Sufficient condition: p is sufficient for q when p is enough: whenever p holds, q holds.

Necessary and sufficient: both directions hold, so $p \leftrightarrow q$.

Forward direction: $P \to Q$. **Backward direction:** $Q \to P$, the converse.

Necessary condition: q is necessary for p when p never happens without q.

KEY IDEAS

1. Same truth value, true; different, false

$P \leftrightarrow Q$ is true exactly when P and Q have the **same** truth value.

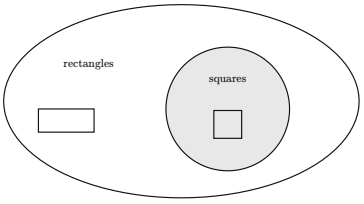
Row	P	Q	$P \leftrightarrow Q$
1	T	T	T
2	T	F	F
3	F	T	F
4	F	F	**T**

The row to notice is the last one. Agreeing on "false" counts as much as agreeing on "true".

$$P \leftrightarrow Q \;\equiv\; (P \to Q) \wedge (Q \to P)$$

A biconditional is two if-then statements at once. Each direction rules out one of the mismatch rows, so you need **both**.

2. Necessary and sufficient



If "if p, then q" is true, the region where p holds sits inside the region where q holds. Every square is a rectangle, so being a square is **sufficient** for being a rectangle, and being a rectangle is **necessary** for being a square.

In "if p, then q": the part after "if" is **sufficient**; the part after "then" is **necessary**.

These all say $p \to q$: "if p, then q", "q if p", "p only if q", "p is sufficient for q", "q is necessary for p".

"If" brings in a sufficient condition and "only if" a necessary one, so "**if and only if**" claims both.

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WATCH OUT

- **One direction is not enough.** "If an integer is divisible by 6, it is even" is true, but "divisible by 6 if and only if even" is false: 4 is even and not divisible by 6.
- **Check every case, not one.** To show a condition is sufficient you need an argument for all cases. To show it is not, one counterexample is enough.
- **The answer depends on the collection.** Over all real numbers, $x^2 = 4$ is not sufficient for $x = 2$. Over positive numbers it is.
- **"Necessary" does not mean "comes first" and "sufficient" does not mean "causes".** The words only say which cases lie inside which.
- **Everyday speech often means more.** "You can go out if you finish your homework" is usually heard as "if and only if". In math, "if" claims one direction only.

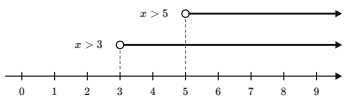
WORKED EXAMPLES

Example 1. Name the row and give the truth value. (a) $\sqrt{49} = 7$ if and only if $7^2 = 49$. (b) $0.5 > 0.6$ if and only if $\frac{1}{2} = 0.5$. (c) $2^{10} < 1000$ if and only if $10^3 < 1000$.

1. (a) Both parts true: row 1.
2. (b) $0.5 > 0.6$ is false, $\frac{1}{2} = 0.5$ is true: row 3.
3. (c) $2^{10} = 1024$, not less than 1000; and $10^3 = 1000$, not less than itself. Both false: row 4.

Answer: (a) true, (b) false, (c) true, since both parts agree on false.

Example 2. x is a real number. Is each condition sufficient, necessary, both or neither for $x > 3$? (a) $x > 5$ (b) $x > 0$ (c) $2x - 6 > 0$ (d) $x \neq 4$



Ask two questions: does it force $x > 3$ (sufficient)? Does $x > 3$ force it (necessary)?

1. (a) $x > 5$ forces $x > 3$. But $x = 4$ has $x > 3$ and not $x > 5$. Sufficient, not necessary.
2. (b) $x = 1$ has $x > 0$ and not $x > 3$, so not sufficient. Every number above 3 is above 0, so necessary.
3. (c) $2x - 6 > 0 \iff 2x > 6 \iff x > 3$. Each step can be undone, so the two hold for exactly the same numbers.
4. (d) $x = 0$ has $x \neq 4$ but not $x > 3$; $x = 4$ has $x > 3$ but not $x \neq 4$.

Answer: (a) sufficient only, (b) necessary only, (c) necessary and sufficient, (d) neither.

Example 3. Is "$x^2 = 4$ if and only if $x = 2$" true for every real x? If not, repair it.

1. At $x = 2$: $2^2 = 4$ is true and $x = 2$ is true. The parts agree.
2. At $x = -2$: $(-2)^2 = 4$ is true but $x = 2$ is false. The parts disagree, so the claim fails.
3. So $x = 2$ is sufficient for $x^2 = 4$ but not necessary.
4. Repair: the solutions of $x^2 = 4$ are exactly 2 and −2.

Answer: false, with $x = -2$ as the counterexample. Repaired: "$x^2 = 4$ if and only if $x = 2$ or $x = -2$", true for every real x.

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