Conditionals: converse, inverse and contrapositive

Almost every theorem says "if this, then that". What exactly does such a sentence promise, and which of the three ways of turning it around still makes the same promise?

WORDS TO KNOW

Conditional: a statement "if P, then Q". It is written $P \to Q$, or $P \Rightarrow Q$.

Hypothesis: the part after "if", here P.

Conclusion: the part after "then", here Q.

Vacuously true: true because the hypothesis is false, so the promise asks nothing.

Logically equivalent ($\equiv$): two statements with the same truth value in every possible case.

KEY IDEAS

1. A conditional is a promise

"If P, then Q" promises that whenever P holds, Q holds too. It is broken in only one case.

P	Q	$P \to Q$
T	T	T
T	F	**F**
F	T	T
F	F	T

$P \to Q$ is false **only** when P is true and Q is false.

The same promise in other words: "P implies Q", "Q if P", "Q whenever P", "P **only if** Q", "P is sufficient for Q", "Q is necessary for P". "Every square is a rectangle" means "if a shape is a square, then it is a rectangle".

2. Three ways to turn it around

Name	Statement	Built by
original	$P \to Q$	
converse	$Q \to P$	swap
inverse	$\neg P \to \neg Q$	negate both
contrapositive	$\neg Q \to \neg P$	swap and negate both

Contrapositive law: $P \to Q \ \equiv \ \neg Q \to \neg P$. They forbid the same case, P without Q.

The converse and the inverse are equivalent to each other, but **not** to the original. A theorem's converse needs its own proof, and it may be false.

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WATCH OUT

- **"Only if" points forward.** "You may board only if you have a ticket" means *if* you board, *then* you have a ticket. It does not say a ticket gets you on board.
- **Do not use the converse.** From "every square is a rectangle" you cannot get "every rectangle is a square": a 2 by 5 rectangle is not a square.
- **The original says nothing when the hypothesis fails.** That is why the inverse does not follow.
- **The contrapositive is not the negation.** The negation of $P \to Q$ is "P and not Q", which always has the **opposite** truth value.
- **Everyday "if" is looser.** In math, "if" goes one way only. The exception is a definition, which is meant both ways.

WORKED EXAMPLES

Example 1. True or false? (a) If 7 is odd, then $7^2 = 49$. (b) If 6 is odd, then $6^2 = 49$. (c) If 2 is even, then 2 is not prime. (d) If 5 > 8, then 5 < 8.

1. (a) Hypothesis true, conclusion true. True.
2. (b) 6 is even, so the hypothesis is false. That settles it: true (vacuously).
3. (c) 2 is even, but 2 is prime, so the conclusion is false. Hypothesis true, conclusion false: false.
4. (d) 5 > 8 is false, so the statement is true, even though the two parts contradict each other.

Answer: only (c) is false. The table asks only which row a statement is in, not whether its parts are connected.

Example 2. For whole numbers n: "if n is a multiple of 6, then n is a multiple of 3." Write the converse, inverse and contrapositive, and decide which of the four are true.

1. Original: true. If $n = 6k$, then $n = 3(2k)$.
2. Converse: if n is a multiple of 3, then n is a multiple of 6. False: 9 is a multiple of 3 but $9 \div 6 = 1.5$.
3. Inverse: if n is not a multiple of 6, then n is not a multiple of 3. False: 9 again.
4. Contrapositive: if n is not a multiple of 3, then n is not a multiple of 6. True, because it is equivalent to the original.

Answer: the original and the contrapositive are true; the converse and the inverse are false, and 9 breaks both.

Example 3. Prove that for every whole number n: if n^2 is even, then n is even.

Plan: the direct start gives a fact about n^2 with no way back to n, so prove the contrapositive instead.

1. Contrapositive: if n is odd, then n^2 is odd.
2. Suppose n is odd, so $n = 2k + 1$ for a whole number k.
3. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is one more than an even number, so odd.
4. The contrapositive holds for every n, so the original does too.

Answer: proved. **Check:** $n = 10$ gives 100, both even; $n = 7$ gives 49, both odd.

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