Conjectures, and why a pattern is not a proof

A guess backed by examples is not yet a fact. What status does it have, what ends the wait, and why can checking more cases never prove it?

WORDS TO KNOW

Conjecture: a precise statement that is believed true but has been neither proved nor disproved.

Statement: a sentence that is either true or false. A question, an instruction or an opinion is not one.

Case: one particular instance of a claim, such as one value of n or one triangle.

Pattern: a rule that fits every case checked *so far*.

Proof: an argument that the rule is right in *every* case, checked or not.

Counterexample: one case where a claim fails.

KEY IDEAS

1. A conjecture is open, and only two things close it

A conjecture must be a statement, it must be precise (every word has one meaning, and it says what collection it is about), and it must be open. It is already true or false while it waits; we just do not know which yet.

State	What puts it there	Its name
open	nothing has settled it yet	a conjecture
proved	an argument that covers every case	a theorem
disproved	a counterexample	a false conjecture

Only a proof or a counterexample changes the status. A thousand agreeing cases, or a million, leave it open.

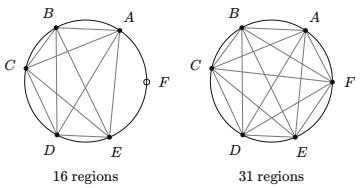
2. One case cuts differently for the two shapes

Most conjectures say "**for every** x, ...". A few say "**there is** an x with ...".

Shape	One example where it works	One case where it fails
for every	settles nothing	disproves it
there is	proves it	settles nothing (unless you have checked every case)

A false conjecture can often be **repaired** by narrowing what it is about, so the failing case is left out. The repaired claim is a new conjecture and needs its own proof.

3. A pattern is not a proof



Mark points on a circle and join every pair with a chord. 1, 2, 3, 4, 5 points give 1, 2, 4, 8, 16 regions. Anyone would guess 32 next. Six points give 31.

Many different rules fit the same five numbers and then go different ways. A list of checked cases is finite; a rule is about every case. What closes the gap is a **reason**: an argument that keeps n as a letter, so the same steps work for every n, including ones nobody has tried.

Examples are how conjectures are found. Only a proof shows they hold every time.

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WATCH OUT

- **More checks do not change the status.** Goldbach's conjecture has been checked for every even number up to 4×10^{18} and is still open.
- **The simplest rule is a habit, not a guarantee.** "Double each time" fits 1, 2, 4, 8, 16 and is wrong for the circle.
- **Vague words make a claim untestable.** If "large" or "usually" is in it, nobody can say what a counterexample would be.
- **Test where it might fail.** A case unlike the ones that suggested the claim is worth more than more of the same: zero, the smallest case, an odd-looking shape.
- **Checking every case is a proof** when there are only finitely many cases. "Every even number from 4 to 30 is a sum of two primes" is settled by checking all fourteen.

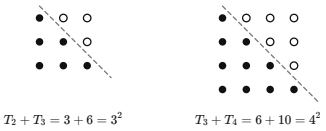
WORKED EXAMPLES

Example 1. Settle each claim. (a) Every prime is odd. (b) There is a prime between 24 and 28.

1. (a) is a for-every claim, so one failing case disproves it. The prime 2 is even.
2. (b) is a there-is claim, so a failed try proves nothing: every candidate must be checked. The candidates are 25, 26 and 27.
3. 25 = 5 × 5, 26 = 2 × 13 and 27 = 3 × 9. None is prime.

Answer: (a) false, with 2 as the counterexample. (b) false, since all three candidates fail.

Example 2. The triangular numbers are $T_n = 1 + 2 + \cdots + n$: 1, 3, 6, 10, 15, ... Find a pattern in $T_n + T_{n+1}$, state it, and prove it.

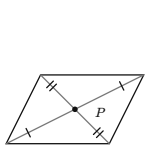


1. Cases: 1 + 3 = 4, 3 + 6 = 9, 6 + 10 = 16, 10 + 15 = 25. These are $2^2, 3^2, 4^2, 5^2$.
2. Conjecture: for every whole number $n \geq 1$, $T_n + T_{n+1} = (n+1)^2$. So far this is only a pattern.
3. Reason: cut a square of $(n+1)$ by $(n+1)$ dots along a diagonal. The two pieces are T_{n+1} and T_n.
4. Algebra, with n kept as a letter, using $T_n = \dfrac{n(n+1)}{2}$:

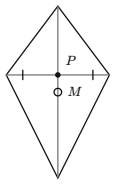
$$T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{(n+1)(2n+2)}{2} = (n+1)^2$$

Answer: $T_n + T_{n+1} = (n+1)^2$ for every $n \geq 1$. It is now a theorem. **Check:** $T_7 + T_8 = 28 + 36 = 64 = 8^2$.

Example 3. In a rectangle, a rhombus and a parallelogram, the diagonals cut each other in half. Test "in every quadrilateral, the diagonals bisect each other" on the kite with corners (0, 6), (−3, 2), (0, −4), (3, 2), and repair it.



P bisects both



$P \neq M$

1. The across diagonal runs from (−3, 2) to (3, 2). Its midpoint is (0, 2).
2. The up-and-down diagonal runs from (0, 6) to (0, −4). Its midpoint is (0, 1).
3. The diagonals cross at (0, 2), which is not the midpoint (0, 1) of the second one. The kite is a counterexample.
4. Repair: narrow the collection to the shapes that passed, and leave the kite out.

Answer: the conjecture is false. Repaired: "in every parallelogram, the diagonals bisect each other", a new claim that needs its own proof.

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