Counterexamples

A claim says something holds for every member of a collection. One member where it fails is enough to show the claim is false. How do you find that member, and how do you show it really fails?

WORDS TO KNOW

Universal claim: a claim about **every** member of a collection. Look for "every", "all", "always", "never".

Hypothesis: the "if" part of an if-then claim.

Counterexample: a member of the domain that makes the hypothesis true and the conclusion false.

Domain: the collection the claim is about, such as all real numbers or all rectangles.

Conclusion: the "then" part of an if-then claim.

KEY IDEAS

1. What counts as a counterexample

Many claims hide an "every" inside an if-then: "if a number is prime, it is odd" means every prime is odd.

A counterexample must:

1. come from the domain,
2. make the hypothesis **true**,
3. make the conclusion **false**.

A member that makes the hypothesis false can never break the claim. The claim makes no promise about it.

2. One is enough

The opposite of "for every x, the claim holds" is "for some x, the claim fails". To show that "for some" is true, you only need to show one x. So one counterexample proves a universal claim false, no matter how many cases worked before.

To refute "for every": show **one** member where it fails.

To prove "for every": you need an argument that covers every member. Checking cases is not enough (unless the domain is finite and you check them all).

3. Where to look, and how to check

Failures hide in the cases the author did not picture.

- **Edge cases:** 0, 1, negative numbers, fractions between 0 and 1, two equal values, the smallest allowed case, a square counted as a rectangle.
- **The structure:** factor the expression, and look for inputs that make every term share a factor.
- **A failed proof:** the step that needs something extra (such as "x is positive") shows where to search.

Then check both halves exactly: the hypothesis holds, and the conclusion fails, with arithmetic anyone can repeat.

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WATCH OUT

- **Breaking the hypothesis does not count.** For "if $a > b$, then $a^2 > b^2$", the pair $a = -3$, $b = 2$ is not a counterexample, because −3 > 2 is false.
- **"It looks false" is not a check.** Write out a factorization or compare two exact numbers. A rounded calculator display is not proof.
- **Many successes prove nothing.** The next case may be the one that fails.
- **A counterexample only kills the claim as written.** It does not say how often the claim fails, and a narrower claim may still be true.
- **It cannot refute a "there is" claim.** "There is an x with ..." needs only one success somewhere, so failures leave it standing. Only showing that every member fails refutes it.

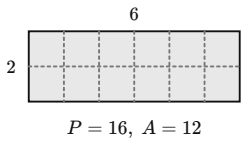
WORKED EXAMPLES

Example 1. Claim: for all real a and b, if $a > b$, then $a^2 > b^2$. Which of (5, 3), (−3, 2) and (1, −2) are counterexamples?

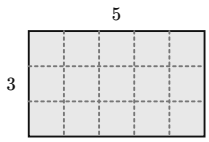
1. (5, 3): 5 > 3 is true and 25 > 9 is true. The claim holds here. Not a counterexample.
2. (−3, 2): −3 > 2 is false, so the hypothesis fails. Not a counterexample.
3. (1, −2): 1 > −2 is true. But $a^2 = 1$ and $b^2 = 4$, and 1 > 4 is false. Hypothesis true, conclusion false.

Answer: only (1, −2) is a counterexample, so the claim is false. The obvious proof needs both numbers positive, which is why the hunt went to negative numbers.

Example 2. Claim: any two rectangles with the same perimeter have the same area.



$P = 16,\ A = 12$



$P = 16,\ A = 15$

1. Keep the perimeter at 16, so length plus width is 8, and change the shape.
2. A 6 by 2 rectangle: perimeter 2(6 + 2) = 16, area 6 × 2 = 12.
3. A 5 by 3 rectangle: perimeter 2(5 + 3) = 16, area 5 × 3 = 15.

Answer: same perimeter, areas 12 and 15. The claim is false.

Example 3. Claim: for every whole number $n \geq 0$, the number $2n^2 + 29$ is prime.

1. Trying $n = 0, 1, 2, \ldots$ gives 29, 31, 37, 47, 61, ..., all prime. A search would need 30 tries to find a failure.
2. Use the structure instead. If n is a multiple of 29, both terms are multiples of 29. Take $n = 29$:
$$2 \times 29^2 + 29 = 29 \times (2 \times 29 + 1) = 29 \times 59 = 1711$$
3. 1711 has the factor 29 and is bigger than 29, so it is not prime.

Answer: $n = 29$ is a counterexample. **Check:** 29 × 59 = 29 × 60 − 29 = 1740 − 29 = 1711, and 2 × 841 + 29 = 1682 + 29 = 1711.

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