Inductive reasoning: conjectures and counterexample figures

A pattern has held in every case you have tried. Does that make it true, how do you hunt for the figure that breaks it, and how do you repair a claim that breaks?

WORDS TO KNOW

Inductive reasoning: going from particular cases to a general statement. You look at examples, see a pattern, and say it always holds.

Deductive reasoning: going from statements already accepted to a conclusion that **must** follow. This is how a conjecture gets proved.

Repair: change a false claim so its counterexamples no longer apply, by adding a hypothesis or weakening the conclusion.

Conjecture: a claim that looks true but has not been proved.

Counterexample figure: one figure that has every property in the hypothesis and lacks the conclusion, both shown exactly.

KEY IDEAS

1. Observe, conjecture, settle

Inductive reasoning makes the guess. Its result is probable, never certain, because the next case has not been checked. Deductive reasoning then decides.

1. **Observe.** Collect cases, as different from each other as you can.
2. **Conjecture.** State the pattern precisely enough that it could be wrong.
3. **Settle.** Find a proof that covers every case, or one figure where it fails.

2. Hunt for the least tidy figure

The figures you draw first are the easy ones: symmetric, upright, acute. They have features the claim never asked for. Remove those features one at a time.

What your drawing has that the claim did not ask for	Break it with
equal sides placed opposite each other	the same sides placed next to each other
two equal sides, a line of symmetry	a triangle with no equal sides
every angle acute	a right angle, then an obtuse one
every corner bulging outward	a dent, such as a dart
a square	a less special shape, such as a rectangle

Put the figure on a grid when you can. Lengths come from the distance formula and angles from slopes, so the failure is exact.

3. Repairing a false claim

- Add a hypothesis, which shrinks the set of figures the claim is about, or weaken the conclusion.
- A repaired claim is a **new** conjecture. It still needs a proof, and that proof must use what was added.
- When you can say *why* the hunt keeps failing, that reason is the start of a proof.

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WATCH OUT

- **Many agreeing cases prove nothing.** Points on a circle give 1, 2, 4, 8, 16 regions, then 31, not 32.
- **A figure that misses a hypothesis proves nothing.** A trapezoid cannot refute a claim about parallelograms, however close it looks.
- **The failure must be exact.** Sides measured as 3.0 cm and 3.1 cm do not make a counterexample. The difference may be the pencil.
- **Leaving out only the figure you found is not a repair.** Every figure like it is still there.
- **Inductive reasoning is not "proof by induction".** Proof by induction is a deductive method that does prove claims about every whole number.

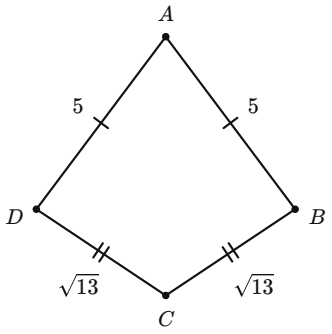
WORKED EXAMPLES

Example 1. Polygons with 3 to 7 sides have 0, 2, 5, 9, 14 diagonals. Predict the number for 8 sides, then give a reason.

1. The differences are 2, 3, 4, 5. If the next is 6, an octagon has 14 + 6 = 20 diagonals. This is a conjecture.
2. Reason: from each corner, diagonals go to every corner except itself and its two neighbors, so each of the n corners starts $n - 3$ diagonals.
3. That counts each diagonal twice, once from each end, so there are $\frac{n(n-3)}{2}$ diagonals.

Answer: 20. **Check:** $\frac{8 \cdot 5}{2} = 20$. The reason holds for every n, so the prediction is now proved.

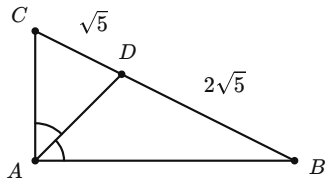
Example 2. Test: "if a quadrilateral has two pairs of equal sides, it is a parallelogram." Use $A(0,4)$, $B(3,0)$, $C(0,-2)$, $D(-3,0)$.



1. First drawings put the equal sides opposite. Put them next to each other instead.
2. Hypothesis: $AB = AD = \sqrt{3^2 + 4^2} = 5$ and $CB = CD = \sqrt{3^2 + 2^2} = \sqrt{13}$. Two pairs of equal sides.
3. Conclusion: in a parallelogram, opposite sides are parallel. $\overline{AB}$ has slope $-\frac{4}{3}$ but the opposite side $\overline{DC}$ has slope $-\frac{2}{3}$. Not parallel.

Answer: $ABCD$ (a kite) is a counterexample, so the claim is false. Repair by saying where the equal sides sit: "a quadrilateral whose **opposite** sides are equal in pairs is a parallelogram."

Example 3. Test: "in every triangle, the bisector of an angle bisects the opposite side." Use $A(0,0)$, $B(6,0)$, $C(0,3)$.



1. First drawings are isosceles. This triangle has no equal sides.
2. The angle at A is a right angle between the axes, so its bisector is the line $y = x$.
3. Side $\overline{BC}$ lies on $x + 2y = 6$. With $y = x$: $3x = 6$, so the bisector meets $\overline{BC}$ at $D(2,2)$.
4. $BD = \sqrt{4^2 + 2^2} = 2\sqrt{5}$ and $DC = \sqrt{2^2 + 1^2} = \sqrt{5}$. These are not equal.

Answer: false, since BD is twice DC. Repair by adding back what the drawings had: "in a triangle with $AB = AC$, the bisector of $\angle A$ bisects $\overline{BC}$."

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