Lines and planes in space

Could two straight lines cross twice? Where two flat surfaces cut through each other, what shape is the place they meet? And in space, must two lines that never meet be parallel?

WORDS TO KNOW

Space: the set of all points. It holds every plane, and points off each one.

Dimension: how many directions you can move in and stay inside the figure. A point is 0, a line 1, a surface 2, a solid 3. They are measured by length (1), area (2) and volume (3).

Parallel: two lines in one plane that never meet. A line and a plane, or two planes, that never meet are parallel too.

Plane figure / space figure: a plane figure has all its points in one plane. A space figure does not.

Intersection: the set of points that lie in both figures. Written $\ell \cap m$. No shared points is written $\varnothing$.

Skew lines: two lines that no plane holds both of. They never meet and are not parallel.

KEY IDEAS

1. Lines and planes meet in a point, a line, or not at all

Two different points fix exactly one line, and a plane is flat: if two different points of a line lie in a plane, the whole line does. Those two facts decide every case.

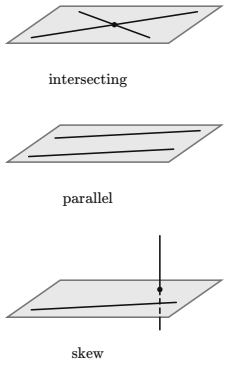
The two figures	How they can meet
Two different lines	in exactly one point, or not at all
A line and a plane	the line lies in the plane, or meets it in exactly one point, or they are parallel
Two different planes	in exactly one line, or not at all (parallel)

Never in two points, three points, or a scattered handful. Two different lines that share two points would be the same line.

These results are about whole lines and planes. Two edges of a box can miss each other while their lines meet further out, so extend to the full line or plane first.

2. Space has a third way for two lines to sit

On flat paper, two lines that never meet are parallel. In space that is not enough. A road running east over a bridge and a road running north under it never meet, but they are not parallel. They are **skew**.



Two different lines in space do exactly one of these:

- **intersect** in one point (they lie in one plane),
- are **parallel** (they lie in one plane and never meet),
- are **skew** (no plane holds both).

Piercing test: if line m meets plane P at only one point X, and line ℓ lies in P but misses X, then ℓ and m are skew.

A line in one plane and a line in a parallel plane are parallel or skew. They cannot meet.

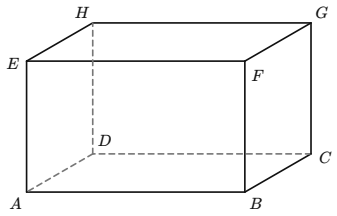
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WATCH OUT

- **"Never meet" is not "parallel" in space.** Parallel lines must also lie in one plane. Lines that never meet and point different ways are skew.
- **Do not trust where lines cross on the page.** The page is flat, so drawn lines cross even when the lines in space do not. Decide from the solid, not the picture.
- **The answer is a whole line, not an edge.** Two faces of a box that share an edge meet in the whole line through that edge.
- **Keep the units straight.** A frame is a length (cm), a surface is an area (cm^2), a solid is a volume (cm^3). None can stand in for another.

WORKED EXAMPLES

Example 1. In the box, name each intersection: (a) plane $ABCD$ and plane $BCGF$; (b) $\overleftrightarrow{AE}$ and plane $EFGH$; (c) plane $ABFE$ and plane $DCGH$.



1. (a) The bottom and the right face share B and C. Two planes that meet share exactly the line through two common points.
2. (b) $\overleftrightarrow{AE}$ does not lie in the top plane (A is below it), so it meets that plane at most once. It meets it at E.
3. (c) The front and back faces stay the same distance apart however far they go, so their planes never meet.

Answer: (a) $\overleftrightarrow{BC}$, the whole line. (b) the point E. (c) $\varnothing$, no points.

Example 2. In the same box, do these pairs intersect, run parallel, or are they skew? (a) $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$ (b) $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$ (c) $\overleftrightarrow{DH}$ and $\overleftrightarrow{EF}$

1. (a) $\overline{AB}$ and $\overline{EF}$ are opposite sides of the rectangle $ABFE$: one plane, no meeting point. Parallel.
2. (b) $\overleftrightarrow{AB}$ lies in the floor and $\overleftrightarrow{FG}$ in the top, which are parallel planes, so the lines are parallel or skew. One runs left to right and the other front to back, so they are not parallel. Skew.
3. (c) $\overleftrightarrow{DH}$ meets the top plane only at H. $\overleftrightarrow{EF}$ lies in that plane and misses H. By the piercing test, skew.

Answer: (a) parallel, (b) skew, (c) skew.

Example 3. Five lines lie in one plane. No two are parallel and no three pass through one point. How many crossing points are there?

1. In one plane, lines that are not parallel meet, and two different lines meet only once. So every pair of lines makes exactly one crossing.
2. No three lines share a point, so no crossing is counted by two pairs. Count the pairs.
3. Each of the 5 lines pairs with 4 others: $5 \times 4 = 20$. That counts every pair twice, so $\frac{5 \times 4}{2} = 10$.

Answer: 10 crossing points.

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