Converses and biconditional definitions

Every reason in a proof is a definition, a postulate or a theorem. Which of them still work when you turn them around, and how do you test a proposed definition?

WORDS TO KNOW

Definition: says what a word means by naming its exact condition. It makes no claim, so it is never proved.

Theorem: a statement proved from definitions, postulates and earlier theorems.

Too loose / too tight: a proposed definition is too loose if it lets in something it should not, and too tight if it shuts out something it should let in.

Postulate (or **axiom**): a statement accepted without proof, such as the segment addition postulate.

Converse: the statement with the hypothesis and conclusion swapped, about the same kind of figure.

KEY IDEAS

1. A theorem does not prove its converse

Turn a theorem around and you get a new claim. It needs its own proof, or one counterexample to sink it: a figure where the conclusion holds and the hypothesis fails.

Theorem	Converse
vertical angles are congruent	false
a linear pair is supplementary	false
if $l \parallel m$, corresponding angles are congruent	true
if $l \parallel m$, alternate interior angles are congruent	true
if $m\angle C = 90^\circ$ in $\triangle ABC$, then $a^2 + b^2 = c^2$	true
if $AB = AC$ in $\triangle ABC$, then $\angle B \cong \angle C$	true
a rectangle's diagonals are congruent	false

When a theorem and its converse are both true, they join into one "if and only if":

$$l \parallel m \iff \text{corresponding angles are congruent}$$

2. A definition works both ways

A definition says the word applies to **exactly** the things that meet the condition. So every definition is a biconditional, even though it is usually written with a plain "if".

$$x \text{ is a } W \iff x \text{ meets the condition}$$

- **Forward** (you are handed the word): it **unpacks** into a fact you can use.
- **Backward** (you have checked the condition): it **earns** you the word.

"Definition of ..." is a valid reason in either direction. A theorem runs only the way it was proved.

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3. Testing a proposed definition

Test both directions.

- **Forward fails** (a real example misses the condition): **too tight**.
- **Backward fails** (something meets the condition but is not the thing): **too loose**.

A true fact about W is not a definition of W if something else has that fact too.

WATCH OUT

- **The contrapositive is not the converse.** "These angles are not congruent, so the lines are not parallel" uses the theorem's contrapositive, which is equivalent to the theorem. It says nothing about the converse.
- **Not every true "if and only if" is a definition.** "Two sides congruent if and only if two angles congruent" links two ideas already defined; each direction needed its own proof. It is a pair of theorems.
- **A drawing cannot supply equal lengths, equal angles, right angles or parallels.** Those must be given, marked or proved.

WORKED EXAMPLES

Example 1. $\overrightarrow{BD}$ bisects $\angle ABC$ when D is inside $\angle ABC$ and $\angle ABD \cong \angle DBC$. (a) $\overrightarrow{BD}$ bisects $\angle ABC$ and $m\angle ABD = 34°$. Find $m\angle ABC$. (b) D is inside $\angle ABC$, $m\angle ABD = (3x+4)°$, $m\angle DBC = (5x-12)°$ and $m\angle ABC = 56°$. Does $\overrightarrow{BD}$ bisect $\angle ABC$?

1. (a) The word is given, so the definition runs **forward**: $m\angle DBC = 34°$ and D is inside the angle.
2. Angle addition: $m\angle ABC = 34° + 34° = 68°$.
3. (b) The word is what we want, so check the condition. By angle addition,
$$(3x+4) + (5x-12) = 56, \qquad 8x - 8 = 56, \qquad x = 8$$
4. Then $m\angle ABD = 3(8) + 4 = 28°$ and $m\angle DBC = 5(8) - 12 = 28°$. Equal, and D is inside, so the definition runs **backward**.

Answer: (a) 68°. (b) Yes, $\overrightarrow{BD}$ bisects $\angle ABC$. Both parts cite "definition of angle bisector".

Example 2. Write the converse of the Vertical Angles Theorem and decide whether it is true.

1. Theorem: if two angles are vertical angles, then they are congruent.
2. Converse: if two angles are congruent, then they are vertical angles.
3. To refute it, find congruent angles that are not vertical. Two perpendicular lines make four right angles. Take two side by side, $\angle 1$ and $\angle 2$: $m\angle 1 = m\angle 2 = 90°$, so $\angle 1 \cong \angle 2$.
4. But $\angle 1$ and $\angle 2$ share a side: they are a linear pair, not a vertical pair.

Answer: the converse is false.

Example 3. Someone proposes: "Two lines are parallel if they do not intersect." Test both directions and repair it.

1. Forward: parallel lines lie in one plane and do not meet, so they do not intersect. This holds.
2. Backward: draw a line across a table top, and hold a pencil level a hand's width above it, pointing across that line at an angle. Their lines never meet, but they run different ways. They are skew, not parallel.
3. So the proposal is too loose. Add what the skew pair lacks.

Answer: too loose. Repaired: "Two lines are parallel if they lie in one plane and do not intersect."

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